УДК 517.98, 517.5

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APPROXIMATION BY INTERPOLATION SPECTRAL SUBSPACES OF OPERATORS WITH DISCRETE SPECTRUM

M. I. Dmytryshyn. *Approximation by interpolation spectral subspaces of operators with discrete spectrum*, Mat. Stud. **55** (2021), 162–170.

The paper describes approximation properties of interpolation spectral subspaces of an unbounded operator A with discrete spectrum $\sigma(A)$ in a Banach space $\mathfrak{X}$, as well as ones corresponding subspaces $\mathcal{E}^{\nu}_{q,p}(A)$ of analytic vectors relative to A. Some properties of subspaces $\mathcal{E}^{\nu}_{q,p}(A)$ are established, including the possibility of their identification with the interpolation subspaces obtained by the real method of interpolation. A relation between spectral subspaces and subspaces $\mathcal{E}^{\nu}_{q,p}(A)$ of analytic vectors of A is also established.

We prove the inequalities that provide a sharp estimate of errors for the best approximations by interpolation spectral subspaces, as well as the subspaces $\mathcal{E}^{\nu}_{q,p}(A)$. Such inequalities fully characterize the subspace of elements from $\mathfrak{X}$ in relation to rapidity of approximations. The obtained estimates of spectral approximation errors are expressed in terms of the quasi-norms of the approximation spaces $\mathcal{B}^{s}_{q,p,\tau}(A)$ associated with the given operator A. In this regard, the E-functional is used that plays a similar role as the module of smoothness in the function theory.

We use the so-called normalization factor to write the constants in the estimates of spectral approximation errors. This normalization factor is determined by the parameters τ and s of the approximation spaces $\mathcal{B}^{s}_{q,p,\tau}(A)$ and has a special form in the case $\tau(1+s)=2$.

Applications to spectral approximations of the regular elliptic operators with variable smooth coefficients in the space $L_q(\Omega)$ over an open bounded set $\Omega\subset\mathbb{R}^n$ and some self-adjoint ordinary elliptic differential operators in a bounded interval $\Omega=(a,b)$ are shown.

1. Introduction. Our purpose is to study the approximation properties of interpolation spectral subspaces relative to a given unbounded operator A with discrete spectrum $\sigma(A)$ in a Banach space $\mathfrak{X}$. We associate the spectral subspaces with the invariant subspaces $\mathcal{E}^{\nu}_{q,p}(A)$ of analytic vectors of A (see [4, 6]). Some necessary to us properties of these subspaces are given in Theorem 1. The relation between $\mathcal{E}^{\nu}_{q,p}(A)$ and spectral subspaces (see Theorem 2) is crucial to obtain a sharp error estimate for the best approximations in $\mathfrak{X}$.

To estimate the best approximation errors, we apply the approximation E-functional (more details in [2, 16]) and the special scale of approximation spaces $\mathcal{B}^{s}_{q,p,\tau}(A)$ associated with A. We give the estimates of spectral approximation errors in terms of the quasi-norms of $\mathcal{B}^{s}_{q,p,\tau}(A)$.

The essential in our approach is that the approximation spaces $\mathcal{B}^{s}_{q,p,\tau}(A)$ can be identified with interpolation spaces obtained by the real method of interpolation.

2010 *Mathematics Subject Classification:* 41A17, 47A58.
Keywords: spectral approximation; exact errors estimates; operator with discrete spectrum.
doi:10.30970/ms.55.2.162-170

Suppose that $(\mathfrak{X}_0, |\cdot|_{\mathfrak{X}_0})$ and $(\mathfrak{X}_1, |\cdot|_{\mathfrak{X}_1})$ are quasi-normed spaces, that form a compatible pair $(\mathfrak{X}_0, \mathfrak{X}_1)$(see e.g. [2, 15]). To explain the K-method, for every compatible pair $(\mathfrak{X}_0, \mathfrak{X}_1)$ we define the K-functional by

$$K(t,x;\mathfrak{X}_0,\mathfrak{X}_1) := \left\{\inf \left(|x_0|^2_{\mathfrak{X}_0} + t^2\,|x_1|^2_{\mathfrak{X}_1}\right)^{1/2} : x_0 \in \mathfrak{X}_0, x_1 \in \mathfrak{X}_1, x_0 + x_1 = x\right\}$$

for $t > 0$ and $x \in \mathfrak{X}_0 + \mathfrak{X}_1$. This definition is the same as in [15]. More usual way is to replace the 2-norm $\left(|x_0|^2_{\mathfrak{X}_0} + t^2\,|x_1|^2_{\mathfrak{X}_1}\right)^{1/2}$ by the 1-norm $|x_0|_{\mathfrak{X}_0} + t\,|x_1|_{\mathfrak{X}_1}$ in this definition, e.g. [2]. But it leads to the same interpolation spaces and equivalent norms. We also consider the functional $K_\infty(t,x;\mathfrak{X}_0,\mathfrak{X}_1) = \inf\limits_{x=x_0+x_1} \max\left(|x_0|_{\mathfrak{X}_0}, t|x_1|_{\mathfrak{X}_1}\right)$.

Now let us define, for every compatible pair $(\mathfrak{X}_0, \mathfrak{X}_1)$, and for $0 < \theta < 1$, $1 \le r \le \infty$,

$$(\mathfrak{X}_0, \mathfrak{X}_1)_{\theta,r} = \left\{x \in \mathfrak{X}_0 + \mathfrak{X}_1 \colon |x|_{(\mathfrak{X}_0,\mathfrak{X}_1)_{\theta,r}} < \infty\right\},$$

this interpolation space with the quasi-norm

$$|x|_{(\mathfrak{X}_0,\mathfrak{X}_1)_{\theta,r}} = \begin{cases} \left(\int_0^\infty \left[t^{-\theta} K(t,x;\mathfrak{X}_0,\mathfrak{X}_1)\right]^r dt/t\right)^{1/r}, & 1 \le r < \infty, \\ \sup_{t>0} t^{-\theta} K(t,x;\mathfrak{X}_0,\mathfrak{X}_1), & r = \infty. \end{cases}$$

Our preferred choice of the 2-norm will be, that is due to use of the so-called normalization factor

$$N_{\theta,r} := \begin{cases} \left(\int_0^\infty t^{r(1-\theta)-1}(1+t^2)^{-r/2} dt\right)^{-1/r}, & 1 \le r < \infty, \\ \theta^{-\theta/2}(1-\theta)^{-(1-\theta)/2}, & r = \infty. \end{cases}$$

This normalization is used in [3], with a focus on establishing when equivalence of norms is in fact equality of norms in the results of the interpolation theory. We use the normalization factor $N_{\theta,r}$ to write the constants in estimates of spectral approximation errors (Theorem 3). The established inequalities fully characterize the subspace of elements from $\mathfrak{X}$ in relation to rapidity of approximations.

Note that exact estimates for approximation errors of spectral approximations for unbounded operators in Banach spaces, using the Besov-type quasi-norms and normalization factor $N'_{\theta,r} = [r\theta(1-\theta)]^{1/r}$ for $1 \le r < \infty$ and $N'_{\theta,\infty} = 1$, are given in [9]. $N_{\theta,r}$ is also used in [5] to study the approximation problem by invariant subspaces of analytic vectors of positive operators in Banach spaces. The calculated constants in estimates are asymptotically exact in the sense that for a fixed θ $(0 < \theta < 1)$ the following limit $\lim\limits_{r\to\infty} (\theta r^2)^{1/r\theta}[N'_{\theta,r}]^{-1/\theta} = 1$ is valid. Actually, in this paper, we also have a view of the exact estimates in the same sense.

Note also that usage of $N_{\theta,r}$ permits to obtain the improved estimates for the spectral approximation errors. In particular, we get the constant $c_{1,\infty} = 1/2$ in the inequality (3) from [5, Theorem 2], while $c_{1,\infty} = 1$ in (1) from [9, Theorem 2].

The last section of this paper contains applications. Similarly to [9], we give new estimates of the spectral approximations errors for a regular elliptic operator in $L_q(\Omega)$ over an open bounded set $\Omega \subset \mathbb{R}^n$ and for some self-adjoint ordinary differential boundary-value problems.

Finally, note that the applications of analytic vectors to approximation problems can be found in [8, 10, 11, 14] and etc. As for exact constants in direct and inverse approximation theorems of the functions theory, see also [1, 17].

2. Subspaces of analytic vectors and spectral subspaces. Let $A\colon \mathfrak{D}^1(A) \to \mathfrak{X}$ be a closed linear operator with a dense domain $\mathfrak{D}^1(A)$ in a Banach space $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$. We assume that A has a discrete spectrum $\sigma(A)$, i.e., its resolvent $R(\lambda, A) = (\lambda - A)^{-1}$ has only isolated eigenvalues $\{\lambda_j \in \mathbb{C}\colon j \in \mathbb{N}\}$ of finite multiplicities, which are poles with the limit at infinity. In particular, this guarantees the compactness of $R(\lambda, A)$ (see e.g. [13, p. 187]).

For any $\nu > 0$ and $k \in \mathbb{Z}_+$ we put $x_{k,\nu} := (A/\nu)^k x$, $x \in \mathfrak{D}^\infty(A) := \bigcap_{k\in\mathbb{Z}_+} \mathfrak{D}^k(A)$. Let $\{x^*_{k,\nu}\}_{k\in\mathbb{Z}_+}$ denotes the rearrangement of the elements by magnitude of the norms:

$$\|x^*_{0,\nu}\|_{\mathfrak{X}} \geq \|x^*_{1,\nu}\|_{\mathfrak{X}} \geq \ldots \geq \|x^*_{k,\nu}\|_{\mathfrak{X}} \geq \ldots.$$

For $1 < q < \infty$ and $1 \leq p \leq \infty$ the subspaces $\mathcal{E}^\nu_{q,p}(A)$ have the following form:

$$\mathcal{E}^\nu_{q,p}(A) = \left\{x \in \mathfrak{X}\colon \|x\|_{\mathcal{E}^\nu_{q,p}(A)} < \infty\right\},$$

where

$$\|x\|_{\mathcal{E}^\nu_{q,p}(A)} = \begin{cases} \left(\sum_{k\in\mathbb{N}} \|x^*_{k-1,\nu}\|^p_{\mathfrak{X}} k^{\frac{p}{q}-1}\right)^{1/p}, & 1 \leq p < \infty, \\ \sup_{k\in\mathbb{N}} \|x^*_{k-1,\nu}\|_{\mathfrak{X}} k^{\frac{1}{q}}, & p = \infty. \end{cases}$$

If $q = p$ then $\mathcal{E}^\nu_{q,q}(A) := \mathcal{E}^\nu_q(A)$ and $\|x\|_{\mathcal{E}^\nu_q(A)} = \left(\sum_{k\in\mathbb{Z}_+} \|x_{k,\nu}\|^q_{\mathfrak{X}}\right)^{1/q}$ in the case $1 \leq q < \infty$ (specified also for $q = \infty$).

Theorem 1. *(a) If $0 < \theta < 1$ and $1 \leq r \leq \infty$ then*

$$(\mathcal{E}^\nu_1(A), \mathcal{E}^\nu_\infty(A))_{\theta,r} = \mathcal{E}^\nu_{1/(1-\theta),r}(A). \tag{1}$$

(b) The contractive inclusion $\mathcal{E}^\nu_{q,p}(A) \hookrightarrow \mathcal{E}^\mu_{q,p}(A)$ with $\mu > \nu$ holds.

(c) The restriction $A|_{\mathcal{E}^\nu_{q,p}(A)}$ is a bounded operator in $\mathcal{E}^\nu_{q,p}(A)$.

(d) Every space $\mathcal{E}^\nu_{q,p}(A)$ is complete.

Proof. (a) As follows from [16, Remark 3.1],

$$K_\infty(t, x; \mathcal{E}^\nu_1(A), \mathcal{E}^\nu_\infty(A)) \leq K(t, x; \mathcal{E}^\nu_1(A), \mathcal{E}^\nu_\infty(A)) \leq \sqrt{2} K_\infty(t, x; \mathcal{E}^\nu_1(A), \mathcal{E}^\nu_\infty(A)). \tag{2}$$

Using [19, Theorem 1.18.3/1] and (2) for $1 \leq r < \infty$, we obtain

$$\|x\|^r_{\left(\mathcal{E}^\nu_1(A), \mathcal{E}^\nu_\infty(A)\right)_{\theta,r}} \sim \sum_{s=1}^\infty s^{-\theta r-1} \left(\sum_{k=1}^{s-1} \|x^*_{k,\nu}\|_{\mathfrak{X}}\right)^r \geq \sum_{s=1}^\infty s^{(1-\theta)r-1} \|x^*_{s-1,\nu}\|^r_{\mathfrak{X}},$$

$$\sum_{s=1}^\infty s^{-\theta r-1} \left(\sum_{k=1}^{s-1} \|x^*_{k,\nu}\|_{\mathfrak{X}}\right)^r \leq c \sum_{k=1}^\infty k^{(1-\theta)r-1} \|x^*_{k-1,\nu}\|^r_{\mathfrak{X}}.$$

Consequently, we get (1) for $1 \leq r < \infty$. In the case $r = \infty$, one obtains

$$\|x\|_{\left(\mathcal{E}^\nu_1(A), \mathcal{E}^\nu_\infty(A)\right)_{\theta,\infty}} \sim \sup_s s^{-\theta} \sum_{k=0}^{s-1} \|x^*_{k,\nu}\|_{\mathfrak{X}} \sim \sup_s s^{1-\theta} \|x^*_{s-1,\nu}\|_{\mathfrak{X}}.$$

(b) For any $\mu > \nu$, we have

$$\|x\|_{\mathcal{E}^\mu_{q,p}(A)} \leq \|x\|_{\mathcal{E}^\nu_{q,p}(A)}, \quad x \in \mathcal{E}^\nu_{q,p}(A),$$

that yields the contractive inclusion $\mathcal{E}^{\nu}_{q,p}(A) \hookrightarrow \mathcal{E}^{\mu}_{q,p}(A)$.

(c) If $x \in \mathcal{E}^{\nu}_{q,p}(A)$ and $1 \le p < \infty$, then

$$\|Ax\|^p_{\mathcal{E}^{\nu}_{q,p}(A)} = \nu^p \sum_{k\in\mathbb{Z}_+} (k+1)^{\frac{p}{q}-1} \left\| (A/\nu)^k x \right\|^p_{\mathfrak{X}} \le \nu^p \|x\|^p_{\mathcal{E}^{\nu}_{q,p}(A)},$$

with the modification when $p = \infty$,

$$\|Ax\|_{\mathcal{E}^{\nu}_{q,\infty}(A)} = \nu \sup_{k\in\mathbb{Z}_+} (k+1)^{1/q} \left\| (A/\nu)^k x \right\|_{\mathfrak{X}} \le \nu \|x\|_{\mathcal{E}^{\nu}_{q,\infty}(A)}.$$

It follows that the invariance and boundedness of $A|_{\mathcal{E}^{\nu}_{q,p}(A)}$ in $\mathcal{E}^{\nu}_{q,p}(A)$.

(d) By [7, Theorem 1(iv)] we have the completeness of $\mathcal{E}^{\nu}_{q}(A)$ for $q = 1, \infty$. Then the space $\mathcal{E}^{\nu}_{q,p}(A)$ is complete as an interpolation space according to (1). □

Let $\mathcal{R}_{\lambda_j}(A) = \{x \in \mathfrak{D}^{\infty}(A) \colon (\lambda_j - A)^{r_j} x = 0\}$ be a spectral subspace, corresponding to the eigenvalue λ_j of multiplicity r_j and $\mathcal{R}^{\nu}(A)$ be the linear span in $\mathfrak{X}$ of all spectral subspaces $\mathcal{R}_{\lambda_j}(A)$ such that $|\lambda_j| < \nu$. Next, let $\mathcal{S}_{\lambda_j}(A) = \{x \in \mathfrak{D}^{\infty}(A) : (\lambda_j - A)x = 0\}$ be a subspace of eigenvectors, corresponding to $\lambda_j \in \sigma(A)$ and $\mathcal{S}^{\nu}(A)$ be the linear span of all $\mathcal{S}_{\lambda_j}(A)$ such that $|\lambda_j| = \nu$, $\lambda_j \in \sigma(A)$. Denote $\mathcal{Q}^{\nu}(A) = \mathcal{R}^{\nu}(A) \oplus \mathcal{S}^{\nu}(A)$ and let us show the relation between the subspaces of analytic vectors and spectral subspaces.

Theorem 2. *The following equalities hold*

$$\mathcal{E}^{\nu}_{q}(A) = \mathcal{R}^{\nu}(A), \quad \mathcal{E}^{\nu}_{\infty}(A) = \mathcal{Q}^{\nu}(A), \tag{3}$$

where $1 \le q < \infty$. *If* $1 < q < \infty$ *and* $1 \le p \le \infty$ *then*

$$\mathcal{E}^{\nu}_{q,p}(A) = (\mathcal{R}^{\nu}(A), \mathcal{Q}^{\nu}(A))_{1-1/q,p}\,. \tag{4}$$

Proof. Each spectral subspace $\mathcal{R}^{\nu}(A)$ coincides with the range of Riesz projector $P_{\nu} = (2\pi i)^{-1} \int_{\gamma} (\lambda - A)^{-1} d\lambda$, where γ is a closed contour, spanning all eigenvalues λ_j of A such that $|\lambda_j| < \nu$ [12, Theorem 5.14.3]. The spectral radius of $AP_{\nu} = A|_{\mathcal{R}^{\nu}(A)}$ is less than ν, i.e. $\lim_{k\to+\infty} \|(AP_{\nu})^k\|^{1/k} < \nu$. So,

$$\|x\|^q_{\mathcal{E}^{\nu}_{q}(A)} = \sum_{k\in\mathbb{Z}_+} \left\|(A/\nu)^k x\right\|^q_{\mathfrak{X}} \le \|x\|^q_{\mathfrak{X}} \sum_{k\in\mathbb{Z}_+} \|AP_{\nu}\|^{kq}/\nu^{kq} < \infty$$

for all $x \in \mathcal{R}^{\nu}(A)$. Thus, $\mathcal{R}^{\nu}(A) \subset \mathcal{E}^{\nu}_{q}(A)$ for any $1 \le q < \infty$.

On the other hand, for each $x \in \mathcal{E}^{\nu}_{q}(A)$, we have $\|(\lambda - A)^{-1}x\|_{\mathcal{E}^{\nu}_{q}(A)} \le \|(\lambda - A)^{-1}\| \|x\|_{\mathcal{E}^{\nu}_{q}(A)}$ and $(\lambda - A)^{-1}(\lambda - A)x = (\lambda - A)(\lambda - A)^{-1}x = x$ for all λ located on the resolvent set $\rho(A)$ of A. Hence, $(\lambda - A|_{\mathcal{E}^{\nu}_{q}(A)})^{-1}$ is the resolvent of $A|_{\mathcal{E}^{\nu}_{q}(A)}$ and $\rho(A) \subset \rho(A|_{\mathcal{E}^{\nu}_{q}(A)})$. So, the unit operator $I|_{\mathcal{E}^{\nu}_{q}(A)}$ on $\mathcal{E}^{\nu}_{q}(A)$ can be represented as $I|_{\mathcal{E}^{\nu}_{q}(A)} = (2\pi i)^{-1} \int_{\gamma} (\lambda - A|_{\mathcal{E}^{\nu}_{q}(A)})^{-1} d\lambda$. It follows that $I|_{\mathcal{E}^{\nu}_{q}(A)} = P_{\nu}|_{\mathcal{E}^{\nu}_{q}(A)}$ and the inclusion $\mathcal{E}^{\nu}_{q}(A) \subset \mathcal{R}^{\nu}(A)$ holds for any $1 \le q < \infty$. So, the first equality (3) is valid.

Using [18, Lemma 1], we have $\mathcal{E}^{\nu}_{\infty}(A) \subset \bigoplus_{j:|\lambda_j|\le\nu} \mathcal{R}_{\lambda_j}(A)$. Then it is sufficient to prove the equality

$$\mathcal{S}_{\lambda_j}(A) = \mathcal{E}^{\nu}_{\infty}(A) \cap \mathcal{R}_{\lambda_j}(A) \tag{5}$$

for indices j with $|\lambda_j| = \nu$. Assume that (5) is not true. Then there exist root vectors $x_0, \dots, x_r$, corresponding to λ_j, such that $|\lambda_j| = \nu$ and $x_r \in \mathcal{E}^{\nu}_{\infty}(A)$, $r \geq 1$. From the equality

$$A^k x_r = \sum_{i=0}^{r} \binom{k}{i} \lambda_j^{k-i} x_{r-i}, \ k \geq r,$$

it follows that

$$\lim_{k\to\infty} \frac{\|A^k x_r\|_{\mathfrak{X}}}{\binom{k}{r}\nu^k} = \nu^{-r}\|x_0\|_{\mathfrak{X}}.$$

Since $x_0 \neq 0$, one has $\nu^{-r}\|x_0\|_{\mathfrak{X}} \neq 0$ and $x_r \notin \mathcal{E}^{\nu}_{\infty}(A)$. So, the equality (5) holds for all j such that $|\lambda_j| = \nu$, as well as the second equality (3) is valid.

The equality (4) directly follows from (1) and (3). □

3. Estimates of spectral approximation errors. We study in this section the case of spectral approximation, where the operator A has a discrete spectrum in a Banach space $\mathfrak{X}$.

Following [6], on the union $\mathcal{E}_{q,p}(A) = \bigcup_{\nu>0} \mathcal{E}^{\nu}_{q,p}(A)$ we define the quasi-norm

$$|x|_{\mathcal{E}_{q,p}(A)} = \|x\|_{\mathfrak{X}} + \inf\left\{\nu > 0 \colon x \in \mathcal{E}^{\nu}_{q,p}(A)\right\},$$

so that $|x+y|_{\mathcal{E}_{q,p}(A)} \leq |x|_{\mathcal{E}_{q,p}(A)} + |y|_{\mathcal{E}_{q,p}(A)}$ for all $x, y \in \mathcal{E}_{q,p}(A)$.

For a pair indices $\{0 < s < \infty, 0 < \tau \leq \infty\}$ or $\{0 \leq s < \infty, \tau = \infty\}$, we assign the approximation spaces $\mathcal{B}^s_{q,p,\tau}(A) = \{x \in \mathfrak{X} : |x|_{\mathcal{B}^s_{q,p,\tau}(A)} < \infty\}$, where

$$|x|_{\mathcal{B}^s_{q,p,\tau}(A)} = \begin{cases} \left(\int_0^\infty \left[t^s E(t, x; \mathcal{E}_{q,p}(A), \mathfrak{X})\right]^\tau dt/t\right)^{1/\tau}, & 0 < \tau < \infty, \\ \sup_{t>0} t^s E(t, x; \mathcal{E}_{q,p}(A), \mathfrak{X}), & \tau = \infty, \end{cases}$$

and $E(t, x; \mathcal{E}_{q,p}(A), \mathfrak{X}) = \inf\left\{\|x - x_0\|_{\mathfrak{X}} \colon x_0 \in \mathcal{E}_{q,p}(A),\ |x_0|_{\mathcal{E}_{q,p}(A)} \leq t\right\}$ for all $x \in \mathfrak{X}$.

If $q = p$ then $\mathcal{E}_{q,q}(A) := \mathcal{E}_q(A)$ and we obtain the approximation spaces $\mathcal{B}^s_{q,q,\tau}(A) =: \mathcal{B}^s_{q,\tau}(A)$, which were considered in [7, 9].

Now let us define, for any $x \in \mathfrak{X}$ and $\nu > 0$,

$$\mathcal{D}^{\nu}_{q,p}(x, A) = \inf\left\{\|x - x_0\|_{\mathfrak{X}} \colon x_0 \in \left(\mathcal{R}^{\nu}(A), \mathcal{Q}^{\nu}(A)\right)_{1-1/q,p}\right\}$$

this is a best approximation of element x by root vectors of interpolation spectral subspace $\left(\mathcal{R}^{\nu}(A), \mathcal{Q}^{\nu}(A)\right)_{1-1/q,p}$ relative to A.

Theorem 3. *The following estimate of spectral approximation errors holds*

$$\mathcal{D}^{\nu}_{q,p}(x, A) \leq c_{s,\tau}\nu^{-s}|x|_{\mathcal{B}^s_{q,p,\tau}(A)}, \quad x \in \mathcal{B}^s_{q,p,\tau}(A), \tag{6}$$

with $c_{s,\tau} = 2^{(1+s)/2}(\tau^2(1+s))^{-1/\tau} N^{(1+s)}_{1/(1+s),\tau(1+s)}$ *if* $0 < \tau < \infty$ *and* $c_{s,\infty} = 1$.

In addition, if $\tau = 2/(1+s)$ *then*

$$\mathcal{D}^{\nu}_{q,p}(x, A) \leq c_s\nu^{-s}|x|_{\mathcal{B}^s_{q,p,2/(1+s)}(A)}, \quad x \in \mathcal{B}^s_{q,p,2/(1+s)}(A), \tag{7}$$

is achieved for $c_s = \left[((1+s)/\pi)\sin(\pi/(1+s))\right]^{(1+s)/2}$.

Proof. Integrating by parts, we similarly to [9] get

$$\int_0^\infty (v^{-\theta} K_\infty(v,x;\mathcal{E}_{q,p}(A),\mathfrak{X}))^r dv/v = -\frac{1}{\theta r}\int_0^\infty K_\infty(v,x;\mathcal{E}_{q,p}(A),\mathfrak{X})^r dv^{-\theta r} =$$
$$= \frac{1}{\theta r}\int_0^\infty v^{-\theta r} dK_\infty(v,x;\mathcal{E}_{q,p}(A),\mathfrak{X})^r = \frac{1}{\theta r}\int_0^\infty (t/E(t,x;\mathcal{E}_{q,p}(A),\mathfrak{X}))^{-\theta r} dt^r =$$
$$= \frac{1}{\theta r^2}\int_0^\infty (t^s E(t,x;\mathcal{E}_{q,p}(A),\mathfrak{X}))^{\theta r} dt/t \quad \text{with} \quad s = 1/\theta - 1.$$

By [16, Remark 3.1], one obtains that

$$K_\infty(t,x;\mathcal{E}_{q,p}(A),\mathfrak{X}) \le K(t,x;\mathcal{E}_{q,p}(A),\mathfrak{X}) \le \sqrt{2} K_\infty(t,x;\mathcal{E}_{q,p}(A),\mathfrak{X}). \tag{8}$$

Using (8), we get

$$\frac{1}{\theta r^2}|x|^{\theta r}_{\mathcal{B}^s_{q,p,\tau}(A)} = \frac{1}{\theta r^2}\int_0^\infty (t^s E(t,x;\mathcal{E}_{q,p}(A),\mathfrak{X}))^{\theta r} dt/t = \int_0^\infty (v^{-\theta} K_\infty(v,x;\mathcal{E}_{q,p}(A),\mathfrak{X}))^r dv/v \le$$
$$\le \int_0^\infty (v^{-\theta} K(v,x;\mathcal{E}_{q,p}(A),\mathfrak{X}))^r dv/v = |x|^r_{(\mathcal{E}_{q,p}(A),\mathfrak{X})_{\theta,r}}.$$

From the right inequality (8) it follows that

$$|x|^r_{(\mathcal{E}_{q,p}(A),\mathfrak{X})_{\theta,r}} \le 2^{r/2}\int_0^\infty (v^{-\theta} K_\infty(v,x;\mathcal{E}_{q,p}(A),\mathfrak{X}))^r dv/v =$$
$$= 2^{r/2}\frac{1}{\theta r^2}\int_0^\infty (t^s E(t,x;\mathcal{E}_{q,p}(A),\mathfrak{X}))^{\theta r} dt/t = 2^{r/2}\frac{1}{\theta r^2}|x|^{\theta r}_{\mathcal{B}^s_{q,p,\tau}(A)}.$$

As a result, from the previous inequalities, we get

$$|x|^r_{(\mathcal{E}_{q,p}(A),\mathfrak{X})_{\theta,r}} \le 2^{r/2}(\theta r^2)^{-1}|x|^{\theta r}_{\mathcal{B}^s_{q,p,\tau}(A)} \le 2^{r/2}|x|^r_{(\mathcal{E}_{q,p}(A),\mathfrak{X})_{\theta,r}} \quad \text{with} \quad \tau = \theta r. \tag{9}$$

Let us define the function $g(v/t) = (v/t)(1+(v/t)^2)^{-1/2}$, $t, v > 0$. By integration of both sides of $g(v/t)K(t,x;\mathcal{E}_{q,p}(A),\mathfrak{X}) \le K(v,x;\mathcal{E}_{q,p}(A),\mathfrak{X})$, we get

$$\Big(\int_0^\infty \big(v^{-\theta} g(v/t)\big)^r \frac{dv}{v}\Big)^{1/r} K(t,x;\mathcal{E}_{q,p}(A),\mathfrak{X}) \le$$
$$\le \Big(\int_0^\infty \big(v^{-\theta} K(v,x;\mathcal{E}_{q,p}(A),\mathfrak{X})\big)^r \frac{dv}{v}\Big)^{1/r} = |x|_{(\mathcal{E}_{q,p}(A),\mathfrak{X})_{\theta,r}}, \quad \int_0^\infty \big(v^{-\theta} g(v/t)\big)^r \frac{dv}{v} = (t^\theta N_{\theta,r})^{-r}.$$

It follows that

$$K(t,x;\mathcal{E}_{q,p}(A),\mathfrak{X}) \le t^\theta N_{\theta,r}|x|_{(\mathcal{E}_{q,p}(A),\mathfrak{X})_{\theta,r}}. \tag{10}$$

We choose $t > 0$ according to [2, Lemma 7.1.2], so that

$$t^s E(t,x;\mathcal{E}_{q,p}(A),\mathfrak{X}))^\theta \le v^{-\theta} K_\infty(v,x;\mathcal{E}_{q,p}(A),\mathfrak{X}). \tag{11}$$

Taking into account (8), (10) and (11), we have

$$v^{1-\theta} E(v,x;\mathcal{E}_{q,p}(A),\mathfrak{X})^\theta \le t^{-\theta} K_\infty(t,x;\mathcal{E}_{q,p}(A),\mathfrak{X}) \le N_{\theta,r}|x|_{(\mathcal{E}_{q,p}(A),\mathfrak{X})_{\theta,r}}.$$

Applying (9), we obtain $v^{1-\theta} E(v,x;\mathcal{E}_{q,p}(A),\mathfrak{X})^\theta \le \sqrt{2}(\theta r^2)^{-1/r} N_{\theta,r}|x|^\theta_{\mathcal{B}^s_{q,p,\tau}(A)}$.

So, if $1 \le r < \infty$, $\tau = \theta r$ and $s = 1/\theta - 1$, one obtains that

$$E(t, x; \mathcal{E}_{q,p}(A), \mathfrak{X}) \le c_{s,\tau} t^{-s} \, |x|_{\mathcal{B}^s_{q,p,\tau}(A)}, \ x \in \mathcal{B}^s_{q,p,\tau}(A), \tag{12}$$

with $c_{s,\tau} = 2^{(1+s)/2}(\tau^2(1+s))^{-1/\tau} N^{(1+s)}_{1/(1+s),\tau(1+s)}$.

If $r = \infty$ then

$$E(t, x; \mathcal{E}_{q,p}(A), \mathfrak{X}) \le t^{-s} |x|_{\mathcal{B}^s_{q,p,\infty}(A)}, \ x \in \mathcal{B}^s_{q,p,\infty}(A). \tag{13}$$

Let us $r(x_0) = \inf\left\{\nu > 0 \colon x_0 \in \mathcal{E}^\nu_{q,p}(A)\right\}$. If $|x_0|_{\mathcal{E}_{q,p}(A)} = r(x_0) + \|x_0\|_{\mathfrak{X}} < \mu$ then $r(x_0) < \mu - \|x_0\|_{\mathfrak{X}}$. Therefore, $x_0 \in \mathcal{E}^\nu_{q,p}(A)$ for all $\nu > 0$ such that $r(x_0) < \nu < \mu - \|x_0\|_{\mathfrak{X}}$.

By Theorem 1(b), we have $\mathcal{E}^\nu_{q,p}(A) \subset \mathcal{E}^\mu_{q,p}(A)$. It yields $x_0 \in \mathcal{E}^\mu_{q,p}(A)$. Hence, for any $\mu > 0$, the following inequality holds

$$\inf\left\{\|x - x_0\|_{\mathfrak{X}} \colon x_0 \in \mathcal{E}^\mu_{q,p}(A)\right\} \le E(\mu, x; \mathcal{E}_{q,p}(A), \mathfrak{X}), \ x \in \mathfrak{X}. \tag{14}$$

By (12), (13) and (14), it follows that

$$\inf\left\{\|x - x_0\|_{\mathfrak{X}} \colon x_0 \in \mathcal{E}^\nu_{q,p}(A)\right\} \le c_{s,\tau} \nu^{-s} |x|_{\mathcal{B}^s_{q,p,\tau}(A)}, \quad x \in \mathcal{B}^s_{q,p,\tau}(A), \tag{15}$$

with $c_{s,\tau} = 2^{(1+s)/2}(\tau^2(1+s))^{-1/\tau} N^{(1+s)}_{1/(1+s),\tau(1+s)}$ if $0 < \tau < \infty$ and $c_{s,\infty} = 1$. Now, taking into account (4), from (15) we obtain (6).

By [15, Exercise B.5], we have $N_{\theta,2} = ((2/\pi)\sin(\pi\theta))^{1/2}$. So, if $\tau = 2/(1+s)$ the estimate (6) yields (7). □

Remark 1. In the case $q = p$, we get the estimate

$$\inf\{\|x - x_0\|_{\mathfrak{X}} \colon x_0 \in \mathcal{R}^\nu(A)\} \le c_{s,\tau} \nu^{-s} |x|_{\mathcal{B}^s_{q,\tau}(A)}, \ x \in \mathcal{B}^s_{q,\tau}(A),$$

with $c_{s,\tau} = 2^{(1+s)/2}(\tau^2(1+s))^{-1/\tau} N^{(1+s)}_{1/(1+s),\tau(1+s)}$ if $0 < \tau < \infty$ and $c_{s,\infty} = 1$.

If $q = p$ and $\tau = 2/(1+s)$ then

$$\inf\{\|x - x_0\|_{\mathfrak{X}} \colon x_0 \in \mathcal{R}^\nu(A)\} \le c_s \nu^{-s} |x|_{\mathcal{B}^s_{q,2/(1+s)}(A)}, \ x \in \mathcal{B}^s_{q,2/(1+s)}(A),$$

with $c_s = [((1+s)/\pi)\sin(\pi/(1+s))]^{(1+s)/2}$.

4. Applications. In this section, we give the estimates of spectral approximation errors for some classes of elliptic differential operators.

Regular elliptic differential operators.

In the space $L_q(\Omega)$ $(1 < q < \infty)$ over an open bounded set $\Omega \subset \mathbb{R}^n$ with infinitely smooth boundary $\partial\Omega$, we consider the closed linear operator A with the domain $W^{2m}_{q,A}(\Omega) = \left\{u \in W^{2m}_q(\Omega) \colon b_j u \,|_{\partial\Omega} = 0, \ j = 1, \dots, m\right\}$ via the regular elliptic system [19, Def. 5.2.1/4]

$$(Au)(\xi) = \sum_{|\alpha| \le 2m} a_\alpha(\xi) D^\alpha u(\xi), \quad a_\alpha \in C^\infty(\bar{\Omega}), \quad \bar{\Omega} = \Omega \cup \partial\Omega,$$

$$(b_j u)(\xi) = \sum_{|\alpha| \le m_j} b_{j,\alpha}(\xi) D^\alpha u(\xi), \quad b_{j,\alpha} \in C^\infty(\partial\Omega), \ j = 1, \dots, m.$$

We assume that $0 \in \rho(A)$ for simplicity. It follows that A has a compact resolvent $R(\lambda, A)$ for any $\lambda \in \rho(A)$, as well as the spectrum $\sigma(A)$ is discrete and is independent on q [19, Sec. 5.4.4].

For $0 < s < \infty$, $1 < q < \infty$, $1 \le \tau \le \infty$, we consider the subspace of the Besov space $B^s_{q,\tau}(\Omega)$, which is associated with A (see [19, Def. 4.2.1/1]),

$$B^s_{q,\tau,A}(\Omega) = \left\{u \in B^s_{q,\tau}(\Omega) \colon b_j A^k u\,|_{\partial\Omega} = 0, j = 1, \dots, m,\ k \in \mathbb{Z}_+\right\}.$$

Theorem 4. *The following inequality holds,*

$$\mathcal{D}^\nu_{q,p}(u, A) \le c_{s,\tau} \nu^{-s} |u|_{B^s_{q,\tau}(\Omega)}, \quad u \in B^s_{q,\tau,A}(\Omega), \tag{16}$$

with $c_{s,\tau} = 2^{(1+s)/2}(\tau^2(1+s))^{-1/\tau} N^{(1+s)}_{1/(1+s),\tau(1+s)}$ *if* $0 < \tau < \infty$ *and* $c_{s,\infty} = 1$.

If $\tau = 2/(1+s)$, *then*

$$\mathcal{D}^\nu_{q,p}(u, A) \le c_s \nu^{-s} |u|_{B^s_{q,2/(1+s)}(\Omega)}, \quad u \in B^s_{q,2/(1+s),A}(\Omega), \tag{17}$$

with $c_s = [((1+s)/\pi)\sin(\pi/(1+s))]^{(1+s)/2}$.

Proof. Using (15) from [9, Theorem 3] and (1), one obtains for every p ($1 \le p \le \infty$) that

$$\mathcal{B}^s_{q,p,\tau}(A) = B^s_{q,\tau,A}(\Omega).$$

Thus, the inequalities (16) and (17) follow directly from (6) and (7). □

Legendre differential operators.

In the space $L_2(\Omega)$, where $\Omega = (a, b)$, $-\infty < a < b < \infty$, we consider the Legendre differential operators

$$A_{m,l} u = (-1)^m \frac{d^m}{d\xi^m}\left(p^l(\xi)\frac{d^m u}{d\xi^m}\right), \quad l = 0, 1, \dots, m, \quad m = 1, 2, \dots$$

with $\mathfrak{D}(A_{m,l}) = \left\{u \in C^\infty(\bar\Omega) \colon u^{(j)}(a) = u^{(j)}(b) = 0, j = 0, \dots, m-l-1\right\}$ for all indices $l = 0, 1, \dots, m-1$, and $\mathfrak{D}(A_{m,m}) = C^\infty(\bar\Omega)$ (see [19, Def. 7.2.1]).

By [19, Theorem 7.4.1], $A_{m,l}$ has a closure $\bar A_{m,l}$ in $L_2(\Omega)$ with the domain $\mathfrak{D}(\bar A_{m,l}) = \left\{u \in W^{2m}_2(\Omega; p^{2l}) \colon u^{(j)}(a) = u^{(j)}(b) = 0, j = 0, \dots, m-l-1\right\}$ for $l = 0, 1, \dots, m-1$, and $\mathfrak{D}(\bar A_{m,m}) = W^{2m}_2(\Omega; p^{2m})$. In addition, $\bar A_{m,l}$ is the operator with discrete spectrum.

Theorem 5. *The following inequality holds*

$$\mathcal{D}^\nu_{q,p}(u, \bar A_{m,l}) \le c_{s,\tau} \nu^{-s} |u|_{B^s_{2,\tau}(\Omega)}, \quad u \in \mathcal{B}^s_{2,p,\tau}(\bar A_{m,l}), \tag{18}$$

with $c_{s,\tau} = 2^{(1+s)/2}(\tau^2(1+s))^{-1/\tau} N^{(1+s)}_{1/(1+s),\tau(1+s)}$ *if* $0 < \tau < \infty$ *and* $c_{s,\infty} = 1$.

If $\tau = 2/(1+s)$ *then*

$$\mathcal{D}^\nu_{q,p}(u, \bar A_{m,l}) \le c_s \nu^{-s} |u|_{B^s_{2,2/(1+s)}(\Omega)}, \quad u \in \mathcal{B}^s_{2,p,2/(1+s)}(\bar A_{m,l}), \tag{19}$$

with $c_s = [((1+s)/\pi)\sin(\pi/(1+s))]^{(1+s)/2}$.

Proof. Using (17), (18) from [9, Theorem 4] and (1), one obtains for every p ($1 \le p \le \infty$) that

$$\mathcal{B}^s_{2,p,\tau}(\bar A_{m,l}) = \Big\{u \in B^s_{2,\tau}(\Omega) \colon (\bar A^k_{m,l} u)^{(j)}(a) = (\bar A^k_{m,l} u)^{(j)}(b) = 0, \\ j = 0, \dots, m-l-1, k \in \mathbb{Z}_+\Big\}$$

for $l = 0, 1, \dots, m-1$, and $\mathcal{B}^s_{2,p,\tau}(\bar A_{m,m}) = B^s_{2,\tau}(\Omega)$. It remains to apply Theorem 3. □

REFERENCES

1. F.G. Abdullayev, V.V. Savchuk, A.L. Shidlich, P. Ozkartepe, *Exact constants in direct and inverse approximation theorems of functions of several variables in the spaces* S^p, Filomat, **33** (2019), №5, 1471–1484. doi:10.2298/FIL1905471A
2. J. Bergh, J. Löfström, Interpolation spaces, Springer-Verlag, Berlin-Heidelberg-New York, 1976.
3. S. Chandler-Wilde, D. Hewett, A. Moiola, *Interpolation of Hilbert and Sobolev spaces: Quantitative estimates and counterexamples*, Mathematika, **61** (2015), №2, 414–443. doi: 10.1112/S0025579314000278
4. M. Dmytryshyn, *Besov-Lorentz-type spaces and best approximations by exponential type vectors*, Int. J. Math. Anal., **9** (2015), 779–786. doi:10.12988/ijma.2015.5233
5. M. Dmytryshyn, *Approximation of positive operators by analytic vectors*, Carpathian Math. Publ., **12** (2020), №2, 412–418. doi: 10.15330/cmp.12.2.412-418
6. M. Dmytryshyn, O. Lopushansky, *Lorentz type spaces of ultrasmooth vectors of closed operators*, Carpathian Math. Publ., **1** (2009), №1, 8–14. (in Ukrainian)
7. M. Dmytryshyn, O. Lopushansky, *Bernstein-Jackson-type inequalities and Besov spaces associated with unbounded operators*, J. Inequal. Appl., **2014** (2014), №105, 1–12. doi: 10.1186/1029-242X-2014-105
8. M. Dmytryshyn, O. Lopushansky, *Spectral approximations of strongly degenerate elliptic differential operators*, Carpathian Math. Publ., **11** (2019), №1, 48–53. doi: 10.15330/cmp.11.1.48-53
9. M. Dmytryshyn, O. Lopushansky, *On Spectral approximations of unbounded operators*, Complex Anal. Oper. Theory, **13** (2019), №8, 3659–3673. doi: 10.1007/s11785-019-00923-0
10. S. Giulini, *Bernstein and Jackson theorems for the Heisenberg group*, J. Austral. Math. Soc., **38** (1985), 241–254. doi: 10.1017/S1446788700023107
11. M.L. Gorbachuk, Ya.I. Hrushka, S.M. Torba, *Direct and inverse theorems in the theory of approximation by the Ritz method*, Ukrainian Math. J., **57** (2005), №5, 751–764. doi: 10.1007/s11253-005-0225-4
12. E. Hille, R.S. Phillips, Functional analysis and semigroups, Providence, Rhode Island: American Mathematical Society, Colloquium Publications, V.31, 1957.
13. T. Kato, Perturbation theory for linear operators, Springer-Verlag, Berlin-Heidelberg-New York, 1980.
14. O. Lopushansky, R. Tłuczek-Pięciak, *Best approximations by increasing invariant subspaces of self-adjoint operators*, Symmetry, **12** (2020), №1918, 1–12. doi:10.3390/sym12111918
15. W. McLean, Strongly elliptic systems and boundary integral equations, Cambridge Univ. Press, Cambridge, 2000.
16. J. Peetre, G. Sparr, *Interpolation of normed abelian groups*, Ann. Mat. Pura ed Appl., **92** (1972), №1, 217–262. doi: 10.1007/BF02417949
17. J. Prestin, V.V. Savchuk, A.L. Shidlich, *Direct and inverse theorems on the approximation of* 2π*-periodic functions by Taylor–Abel–Poisson operators*, Ukrainian Math. J., **69** (2017), №5, 766–781. doi: 10.1007/s11253-017-1394-7
18. G.V. Radzievskii, *On the best approximations and rate of convergence of decompositions in the root vectors of an operator*, Ukrainian Math. J., **49** (1997), №6, 844–864. doi: 10.1007/BF02513425
19. H. Triebel, Interpolation theory. Function spaces. Differential operators, North-Holland Publishing Company, Amsterdam-New York-Oxford, 1978.

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Received 03.01.2021