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A NOTE ON THE VALUE DISTRIBUTION OF $\varphi f^2 f^{(k)} - 1$

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In this paper, we study the value distribution of the differential polynomial $\varphi f^2 f^{(k)} - 1$, where $f(z)$ is a transcendental meromorphic function, $\varphi(z)$ ($\not\equiv 0$) is a small function of $f(z)$ and k (≥ 2) is a positive integer. We obtain an inequality concerning the Nevanlinna characteristic function $T(r, f)$ estimated by reduced counting function only. Our result extends the result due to J.F. Xu and H.X. Yi [J. Math. Inequal., 10 (2016), 971-976].

1. Introduction, definitions and results. In this paper, by meromorphic function we shall always mean meromorphic function in the complex plane. We shall use the standard notations of the Nevanlinna theory of meromorphic functions as explained in [3], [5], [11] and [12]. For a nonconstant meromorphic function $f(z)$, we denote by $T(r, f)$ the Nevanlinna characteristic function of $f(z)$ and by $S(r, f)$ any quantity satisfying $S(r, f) = o\{T(r, f)\}$ for all r outside a possible exceptional set of finite logarithmic measure. The meromorphic function $\varphi(z)$ is called a small function of $f(z)$, if $T(r, \varphi) = S(r, f)$.

In this research work, we need the following definitions.

Definition 1 ([12])**.** Let $f(z)$ be a nonconstant meromorphic function and p be a positive integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$, we denote by $N_{p)}(r, \frac{1}{f-a})$ *the counting function* of those zeros of $f(z) - a$ (counted with proper multiplicities) whose *multiplicities are not greater* than p and by $\overline{N}_{p)}(r, \frac{1}{f-a})$ the corresponding *reduced counting function.* We denote by $N_{(p+1}(r, \frac{1}{f-a})$ *the counting function* of those zeros of $f(z) - a$ (counted with proper multiplicities) whose *multiplicities are greater than* p and by $\overline{N}_{(p+1}(r, \frac{1}{f-a})$ the corresponding *reduced counting function.* Also we denote by $N_p(r, \frac{1}{f-a})$ the counting function of zeros of $f(z) - a$ with multiplicity exactly p.

Furthermore, we denote by $N_{\neq p}(r, \frac{1}{f-a})$ the counting function of all zeros of $f(z) - a$ (counted with proper multiplicities) except the zeros whose multiplicities are exactly p and by $\overline{N}_{\neq p}(r, \frac{1}{f-a})$ the corresponding reduced counting function.

Definition 2 ([12])**.** Suppose that $f(z)$ is a nonconstant meromorphic function in the complex plane $\mathbb{C}$, and $\alpha(z)$ is a small function of $f(z)$. Let $n_0, n_1, \ldots, n_k$ be non-negative integers. We call $M(f) = \alpha f^{n_0}(f')^{n_1} \cdot \ldots \cdot (f^{(k)})^{n_k}$ a *differential monomial* in f and $n = \sum_{j=0}^{k} n_j$, the degree of $M(f)$. Also let $M_1(f)$, $M_2(f)$, ..., $M_k(f)$ be differential monomials in f of degrees $m_1, m_2, \ldots, m_k$, respectively. Then $P(f) = \sum_{j=1}^{k} M_j(f)$ is said to be a *differential polynomial* in f and $m = \max\{m_1, m_2, \ldots, m_k\}$, *the degree* of $P(f)$.

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Many research works have been done on the value distribution of differential polynomials of meromorphic functions by many mathematicians worldwide (see [2], [4], [7], [9]). In 1979, E. Mues [6] first proved a qualitative result of value distribution for a transcendental meromorphic function $f(z)$ in the complex plane. The result is as follows:

Theorem A. *Let $f(z)$ be a transcendental meromorphic function in the complex plane. Then $f^2 f' - 1$ has infinitely many zeros.*

Naturally, one may ask the following question.

Question 1. What is the quantitative result of Theorem A?

In 1992, Q. D. Zhang ([13]) find out a quantitative result of Theorem A which is as follows.

Theorem B. *Let $f(z)$ be a transcendental meromorphic function in the complex plane. Then*

$$T(r, f) < 6N\Big(r, \frac{1}{f^2 f' - 1}\Big) + S(r, f).$$

In 2011, J. F. Xu, H. X. Yi and Z. L. Zhang ([10]) improved Theorem B by estimating the reduced counting function and proved the following result.

Theorem C. *Let $f(z)$ be a transcendental meromorphic function in the complex plane. Then*

$$T(r, f) \leq 6\overline{N}\Big(r, \frac{1}{f^2 f' - 1}\Big) + S(r, f).$$

Regarding Theorems B and C , one more question arises.

Question 2. What happens if we use a small function instead of a constant in the counting function in Theorems B and C?

In 1992, Q. D. Zhang ([14]) studied the value distribution related to small functions used in counting function and proved the following result.

Theorem D. *Let $f(z)$ be a transcendental meromorphic function in the complex plane and $\varphi(z)$ $(\not\equiv 0)$ be a small function of $f(z)$. Then*

$$T(r, f) \leq 6N\Big(r, \frac{1}{\varphi f^2 f' - 1}\Big) + S(r, f).$$

In 2016, J. F. Xu and H. X. Yi ([8]) improved Theorem D by considering the reduced counting function and proved the following result.

Theorem E. *Let $f(z)$ be a transcendental meromorphic function in the complex plane and $\varphi(z)$ $(\not\equiv 0)$ be a small function of $f(z)$. Then*

$$T(r, f) \leq 6\overline{N}\Big(r, \frac{1}{\varphi f^2 f' - 1}\Big) + S(r, f).$$

Note 1. In [8, 14] the authors had assumed that the set of zeros and poles of $f(z)$ and that of $\varphi(z)$ are disjoint, though they did not mention it in the statement of their main results.

Now it is natural to ask the following question which motivated us to write this paper.

Question 3. What will be the result if we replace the differential polynomial $\varphi f^2 f' - 1$ by $\varphi f^2 f^{(k)} - 1$, where k (≥ 2) is an integer?

In this paper we investigate to find out the possible answer to the above question and obtain the following theorem.

Theorem 1. *Let $f(z)$ be a transcendental meromorphic function in the complex plane, k (≥ 2) be an integer and $\varphi(z)$ ($\not\equiv 0$) be a small function of $f(z)$ such that the set of zeros and poles of $f(z)$ and that of $\varphi(z)$ are disjoint and $\varphi(z)$ has no zeros of multiplicity 2. Then*
$$T(r,f) \leq 6\overline{N}\Big(r, \frac{1}{\varphi f^2 f^{(k)} - 1}\Big) + S(r,f).$$

2. Lemmas. We now state some lemmas which will be needed in the sequel.

Lemma 1 ([3,5,12])**.** *Let $f(z)$ be a transcendental meromorphic function and k be a positive integer. Then*
$$m\Big(r, \frac{f^{(k)}}{f}\Big) = S(r,f).$$

Lemma 2. *Let $f(z)$ be a transcendental meromorphic function, k (≥ 2) be a positive integer and $\varphi(z)$ ($\not\equiv 0$) be a small function of $f(z)$. Then $\varphi f^2 f^{(k)}$ is not equivalently constant.*

Proof. To prove this lemma we follow [8]. Suppose that $\varphi f^2 f^{(k)} \equiv C$, where C is a constant. Obviously, $C \neq 0$. Hence we have
$$\frac{1}{f^3} \equiv \frac{\varphi}{C}\frac{f^{(k)}}{f}, \quad \frac{1}{f^2 f^{(k)}} \equiv \frac{\varphi}{C}.$$
Therefore, using Lemma 1 we get
$$m\Big(r, \frac{1}{f}\Big) \leq \frac{1}{3}m\Big(r, \frac{\varphi}{C}\frac{f^{(k)}}{f}\Big) \leq \frac{1}{3}m\Big(r, \frac{\varphi}{C}\Big) + \frac{1}{3}m\Big(r, \frac{f^{(k)}}{f}\Big) + O(1) = S(r,f),$$
$$N\Big(r, \frac{1}{f}\Big) \leq N\Big(r, \frac{1}{f^2 f^{(k)}}\Big) \leq N\Big(r, \frac{\varphi}{C}\Big) = S(r,f).$$
Therefore, we obtain $T(r,f) = S(r,f)$, a contradiction. Hence $\varphi f^2 f^{(k)}$ is not equivalently constant. □

Lemma 3. *Let $f(z)$ be a transcendental meromorphic function, k (≥ 2) be a positive integer and $\varphi(z)$ ($\not\equiv 0$) be a small function of $f(z)$. Then*
$$3T(r,f) \leq \overline{N}(r,f) + \overline{N}\Big(r, \frac{1}{f}\Big) + N_{k)}\Big(r, \frac{1}{f}\Big) + k\overline{N}_{(k+1}\Big(r, \frac{1}{f}\Big) + \\ + \overline{N}\Big(r, \frac{1}{\varphi f^2 f^{(k)} - 1}\Big) - N_0\Big(r, \frac{1}{(\varphi f^2 f^{(k)})'}\Big) + S(r,f), \tag{1}$$
$$\{N(r,f) - \overline{N}(r,f)\} + \Big\{N\Big(r, \frac{1}{f}\Big) - \overline{N}\Big(r, \frac{1}{f}\Big)\Big\} + \Big\{N_{(k+1}\Big(r, \frac{1}{f}\Big) - k\overline{N}_{(k+1}\Big(r, \frac{1}{f}\Big)\Big\} + \\ + m(r,f) + 2m\Big(r, \frac{1}{f}\Big) \leq \overline{N}\Big(r, \frac{1}{\varphi f^2 f^{(k)} - 1}\Big) - N_0\Big(r, \frac{1}{(\varphi f^2 f^{(k)})'}\Big) + S(r,f), \tag{2}$$
where $N_0(r, \frac{1}{(\varphi f^2 f^{(k)})'})$ denotes the counting function of those zeros of $(\varphi f^2 f^{(k)})'$ which are not the zeros of $f(\varphi f^2 f^{(k)} - 1)$.

Proof. To prove this we follow [8,10]. From Lemma 2, we get $\varphi f^2 f^{(k)} - 1$ is not equivalently constant. Set

$$\frac{1}{f^3} = \frac{\varphi f^{(k)}}{f} - \frac{(\varphi f^2 f^{(k)})'}{f^3}\frac{(\varphi f^2 f^{(k)} - 1)}{(\varphi f^2 f^{(k)})'}.$$

Hence, applying the inequalities $m(r, f+g) \leq m(r, f) + m(r, g) + \log 2$ and $m(r, fg) \leq m(r, f) + m(r, g)$ as explained in [3, p.5], we have from above and Lemma 1

$$3m\Big(r, \frac{1}{f}\Big) = m\Big(r, \frac{1}{f^3}\Big) \leq m\Big(r, \frac{\varphi f^{(k)}}{f}\Big) + m\Big(r, \frac{(\varphi f^2 f^{(k)})'}{f^3}\Big) + m\Big(r, \frac{\varphi f^2 f^{(k)} - 1}{(\varphi f^2 f^{(k)})'}\Big) + O(1) \leq$$
$$\leq m\Big(r, \frac{\varphi f^2 f^{(k)} - 1}{(\varphi f^2 f^{(k)})'}\Big) + S(r, f)$$

Using the first fundamental theorem of Nevanlinna one has

$$m(r, f/f') = T(r, f/f') - N(r, f/f') \leq T(r, f'/f) - N(r, f/f') + O(1) \leq$$
$$\leq N(r, f'/f) - N(r, f/f') + S(r, f).$$

Hence by Lemma 1

$$3m\Big(r, \frac{1}{f}\Big) \leq N\Big(r, \frac{(\varphi f^2 f^{(k)})'}{\varphi f^2 f^{(k)} - 1}\Big) - N\Big(r, \frac{\varphi f^2 f^{(k)} - 1}{(\varphi f^2 f^{(k)})'}\Big) + S(r, f) = N(r, (\varphi f^2 f^{(k)})') +$$
$$+ N\Big(r, \frac{1}{\varphi f^2 f^{(k)} - 1}\Big) - N(r, (\varphi f^2 f^{(k)} - 1)) - N\Big(r, \frac{1}{(\varphi f^2 f^{(k)})'}\Big) + S(r, f),$$

because zeros and poles come from both of $(\varphi f^2 f^{(k)})'$ and $\varphi f^2 f^{(k)} - 1$ letting equal sign particularly of the formula $N(r, fg) \leq N(r, f) + N(r, g)$ as explained in [3, p.5].

By routine observation we see that at a pole of $\varphi f^2 f^{(k)} - 1$ of order l, $(\varphi f^2 f^{(k)})'$ has a pole of order $l+1$. Such poles come from the poles of $f(z)$ and the poles of $\varphi(z)$ only. So

$$N(r, (\varphi f^2 f^{(k)})') - N(r, (\varphi f^2 f^{(k)} - 1)) \leq \overline{N}(r, \varphi f^2 f^{(k)} - 1) \leq$$
$$\leq \overline{N}(r, f) + \overline{N}(r, \varphi) \leq \overline{N}(r, f) + S(r, f).$$

Therefore we have

$$3m\Big(r, \frac{1}{f}\Big) \leq \overline{N}(r, f) + N\Big(r, \frac{1}{\varphi f^2 f^{(k)} - 1}\Big) - N\Big(r, \frac{1}{(\varphi f^2 f^{(k)})'}\Big) + S(r, f).$$

Hence

$$3T(r, f) = 3m\Big(r, \frac{1}{f}\Big) + 3N\Big(r, \frac{1}{f}\Big) + O(1) \leq \overline{N}(r, f) + 3N\Big(r, \frac{1}{f}\Big) +$$
$$+ N\Big(r, \frac{1}{\varphi f^2 f^{(k)} - 1}\Big) - N\Big(r, \frac{1}{(\varphi f^2 f^{(k)})'}\Big) + S(r, f). \tag{3}$$

Let

$$N\Big(r, \frac{1}{(\varphi f^2 f^{(k)})'}\Big) = N_{000}\Big(r, \frac{1}{(\varphi f^2 f^{(k)})'}\Big) + N_{00}\Big(r, \frac{1}{(\varphi f^2 f^{(k)})'}\Big) +$$
$$+ N_0\Big(r, \frac{1}{(\varphi f^2 f^{(k)})'}\Big) + S(r, f), \tag{4}$$

where $N_{000}(r, \frac{1}{(\varphi f^2 f^{(k)})'})$ denotes the counting function of those zeros of $(\varphi f^2 f^{(k)})'$ which come from the zeros of $(\varphi f^2 f^{(k)} - 1)$ and $N_{00}(r, \frac{1}{(\varphi f^2 f^{(k)})'})$ denotes the counting function of those zeros of $(\varphi f^2 f^{(k)})'$ which come from the zeros of $f(z)$. Hence we have

$$N(r, \frac{1}{\varphi f^2 f^{(k)} - 1}) - N_{000}\Big(r, \frac{1}{(\varphi f^2 f^{(k)})'}\Big) = \overline{N}\Big(r, \frac{1}{\varphi f^2 f^{(k)} - 1}\Big). \tag{5}$$

Let z_0 be a zero of $f(z)$ with multiplicity p and pole of $\varphi(z)$ with multiplicity t. We consider the following two cases: **Case 1** and **Case 2**.

Case 1. Let $p \le k$. If $t \le 2p - 1$, then z_0 is a zero of $(\varphi f^2 f^{(k)})'$ with multiplicity at least $2p - t - 1$. If $t \ge 2p$, then z_0 is not the zero of $(\varphi f^2 f^{(k)})'$. Hence in this case the zeros of $(\varphi f^2 f^{(k)})'$ come only from the zeros of $f(z)$ with multiplicity not greater than k and the poles of $\varphi(z)$ with multiplicity at most $2p - 1$.

Case 2. Let $p \ge k + 1$. If $t \le 3p - k - 1$, then z_0 is a zero of $(\varphi f^2 f^{(k)})'$ with multiplicity at least $3p - t - k - 1$. If $t \ge 3p - k$, then z_0 is not the zero of $(\varphi f^2 f^{(k)})'$. Hence in this case the zeros of $(\varphi f^2 f^{(k)})'$ come only from the zeros of $f(z)$ with multiplicity greater than k and the poles of $\varphi(z)$ with multiplicity at most $3p - k - 1$.

Hence we can write

$$N_{00}\Big(r, \frac{1}{(\varphi f^2 f^{(k)})'}\Big) \ge 2N_{k)}\Big(r, \frac{1}{f}\Big) - \overline{N}_{k)}\Big(r, \frac{1}{f}\Big) + 3N_{(k+1}\Big(r, \frac{1}{f}\Big) - (k+1)\overline{N}_{(k+1}\Big(r, \frac{1}{f}\Big) -$$
$$- s\overline{N}(r, \varphi) = 2N\Big(r, \frac{1}{f}\Big) + N_{(k+1}\Big(r, \frac{1}{f}\Big) - \overline{N}_{k)}\Big(r, \frac{1}{f}\Big) - (k+1)\overline{N}_{(k+1}\Big(r, \frac{1}{f}\Big) - s\overline{N}(r, \varphi),$$

where $s = \max\{2p - 1,\ 3p - k - 1\}$. Therefore, we have

$$3N\Big(r, \frac{1}{f}\Big) - N_{00}\Big(r, \frac{1}{(\varphi f^2 f^{(k)})'}\Big) \le N_{k)}\Big(r, \frac{1}{f}\Big) + \overline{N}\Big(r, \frac{1}{f}\Big) + k\overline{N}_{(k+1}\Big(r, \frac{1}{f}\Big) + S(r, f). \tag{6}$$

Combining (3)–(6), we have

$$3T(r, f) \le \overline{N}(r, f) + \overline{N}\Big(r, \frac{1}{f}\Big) + N_{k)}\Big(r, \frac{1}{f}\Big) + k\overline{N}_{(k+1}\Big(r, \frac{1}{f}\Big) +$$
$$+ \overline{N}\Big(r, \frac{1}{\varphi f^2 f^{(k)} - 1}\Big) - N_0\Big(r, \frac{1}{(\varphi f^2 f^{(k)})'}\Big) + S(r, f).$$

Thus the inequality (1) is proved. Since

$$3T(r, f) = N(r, f) + N\Big(r, \frac{1}{f}\Big) + N_{k)}\Big(r, \frac{1}{f}\Big) + N_{(k+1}\Big(r, \frac{1}{f}\Big) + m(r, f) + 2m\Big(r, \frac{1}{f}\Big) + O(1),$$

the inequality (2) can be obtained easily. □

Lemma 4. *Let $f(z)$ be a transcendental meromorphic function and $\varphi(z)$ ($\not\equiv 0$) be a small function of $f(z)$ such that the set of zeros and poles of $f(z)$ and that of $\varphi(z)$ are disjoint. Suppose that $F = \varphi f^2 f^{(k)} - 1$ and $h = \frac{F'}{f}$, where k (≥ 2) is an integer. Let*

$$G(z) = a_1\Big(\frac{F'(z)}{F(z)}\Big)^2 + a_2\Big(\frac{F'(z)}{F(z)}\Big)' + a_3\frac{F'(z)}{F(z)}\frac{h'(z)}{h(z)} + a_4\Big(\frac{h'(z)}{h(z)}\Big)^2 + a_5\Big(\frac{h'(z)}{h(z)}\Big)' +$$

$$+a_6\frac{\varphi'(z)}{\varphi(z)}\frac{F'(z)}{F(z)} + a_7\frac{\varphi'(z)}{\varphi(z)}\frac{h'(z)}{h(z)} + a_8\Big(\frac{\varphi'(z)}{\varphi(z)}\Big)^2 + a_9\Big(\frac{\varphi'(z)}{\varphi(z)}\Big)', \tag{7}$$

where

$$a_1 = 2(k+1)^2 - \frac{(3k+7)(k^2-4k-29)}{k+3},$$
$$a_2 = -(k+5)(k^2-4k-29),\ a_3 = 4(k^2-4k-29),$$
$$a_4 = -4(k+1)(k+3),\ a_5 = 2(k+1)(k+3)(k+5),\ a_6 = -2(k^2-4k-29),$$
$$a_7 = 4(k+1)(k+3),\ a_8 = -(k+1)(k+3),\ a_9 = -(k+1)(k+3)(k+5).$$

If $G(z) \not\equiv 0$, then the simple poles of $f(z)$ are the zeros of $G(z)$.

Proof. Suppose z_0 is a simple pole of $f(z)$. Let

$$f(z) = \frac{a}{(z-z_0)}\Big[1 + b_0(z-z_0) + b_1(z-z_0)^2 + O((z-z_0)^3)\Big], \tag{8}$$

where $a\ (\neq 0),\ b_0,\ b_1$ are constants. Noting that $\varphi(z_0) \neq 0,\ \infty$, let

$$\varphi(z) = c\Big[1 + c_0(z-z_0) + c_1(z-z_0)^2 + O((z-z_0)^3)\Big], \tag{9}$$

where $c\ (\neq 0),\ c_0,\ c_1$ are constants. Using (8) and (9), we obtain the following:

$$\frac{\varphi'(z)}{\varphi(z)} = \Big[c_0 + (2c_1 - c_0^2)(z-z_0) + O((z-z_0)^2)\Big], \tag{10}$$

$$f^2(z) = \frac{a^2}{(z-z_0)^2}\Big[1 + 2b_0(z-z_0) + (b_0^2 + 2b_1)(z-z_0)^2 + O((z-z_0)^3)\Big], \tag{11}$$

$$f^{(k)}(z) = \frac{(-1)^k k! a}{(z-z_0)^{k+1}}\Big[1 + O((z-z_0)^{k+1})\Big]. \tag{12}$$

Using (9), (11) and (12) we get

$$F(z) = \varphi f^2 f^{(k)} - 1 = \frac{(-1)^k k! a^3 c}{(z-z_0)^{k+3}}\Big[1 + (2b_0 + c_0)(z-z_0) +$$
$$+(b_0^2 + 2b_0c_0 + 2b_1 + c_1)(z-z_0)^2 + O((z-z_0)^3)\Big],$$
$$F'(z) = \frac{(-1)^{k+1}(k+3)k! a^3 c}{(z-z_0)^{k+4}}\Big[1 + \frac{k+2}{k+3}\Big(2b_0 + c_0\Big)(z-z_0) +$$
$$+\frac{k+1}{k+3}\Big(b_0^2 + 2b_0c_0 + 2b_1 + c_1\Big)(z-z_0)^2 + O((z-z_0)^3)\Big].$$

Hence,

$$\frac{F'(z)}{F(z)} = -\frac{(k+3)}{(z-z_0)}\Big[1 - \frac{1}{k+3}\Big(2b_0 + c_0\Big)(z-z_0) +$$
$$+\frac{1}{k+3}\Big(2b_0^2 + c_0^2 - 4b_1 - 2c_1\Big)(z-z_0)^2 + O((z-z_0)^3)\Big], \tag{13}$$
$$h(z) = \frac{F'(z)}{f(z)} = \frac{(-1)^{k+1}(k+3)k! a^2 c}{(z-z_0)^{k+3}}\Big[1 + \frac{1}{k+3}\Big\{(k+1)b_0 + (k+2)c_0\Big\}(z-z_0) +$$

$$+\frac{1}{k+3}\Big\{kb_0c_0+(k-1)b_1+(k+1)c_1\Big\}(z-z_0)^2+O((z-z_0)^3)\Big],$$

$$\frac{h'(z)}{h(z)}=-\frac{k+3}{(z-z_0)}\Big[1-\frac{1}{(k+3)^2}\Big\{(k+1)b_0+(k+2)c_0\Big\}(z-z_0)+$$
$$+\frac{1}{(k+3)^3}\Big\{(k+1)^2b_0^2+(k+2)^2c_0^2+4b_0c_0-2(k-1)(k+3)b_1-$$
$$-2(k+1)(k+3)c_1\Big\}(z-z_0)^2+O((z-z_0)^3)\Big]. \tag{14}$$

Thus from (10), (13) and (14) we obtain the following asymptotic relations:

$$\Big(\frac{F'(z)}{F(z)}\Big)^2=\frac{1}{(z-z_0)^2}\Big[(k+3)^2-2(k+3)(2b_0+c_0)(z-z_0)+\Big\{4(k+4)b_0^2+$$
$$+(2k+7)c_0^2+4b_0c_0-8(k+3)b_1-4(k+3)c_1\Big\}(z-z_0)^2+O((z-z_0)^3)\Big], \tag{15}$$

$$\Big(\frac{F'(z)}{F(z)}\Big)'=\frac{1}{(z-z_0)^2}\Big[(k+3)-(2b_0^2+c_0^2-4b_1-2c_1)(z-z_0)^2+O((z-z_0)^3)\Big], \tag{16}$$

$$\frac{F'(z)}{F(z)}\frac{h'(z)}{h(z)}=\frac{1}{(z-z_0)^2}\Big[(k+3)^2-\Big\{(3k+7)b_0+(2k+5)c_0\Big\}(z-z_0)+\Big\{(3k+7)b_0^2+$$
$$+(2k+5)c_0^2+3b_0c_0-2(3k+5)b_1-4(k+2)c_1\Big\}(z-z_0)^2+O((z-z_0)^3)\Big], \tag{17}$$

$$\Big(\frac{h'(z)}{h(z)}\Big)^2=\frac{1}{(z-z_0)^2}\Big[(k+3)^2-2\Big\{(k+1)b_0+(k+2)c_0\Big\}(z-z_0)+$$
$$+\frac{1}{(k+3)^2}\Big\{(k+1)^2(2k+7)b_0^2+(k+2)^2(2k+7)c_0^2+2(k^2+7k+14)b_0c_0-$$
$$-4(k-1)(k+3)^2b_1-4(k+1)(k+3)^2c_1\Big\}(z-z_0)^2+O((z-z_0)^3)\Big], \tag{18}$$

$$\Big(\frac{h'(z)}{h(z)}\Big)'=\frac{1}{(z-z_0)^2}\Big[k+3-\frac{1}{(k+3)^2}\Big\{(k+1)^2b_0^2+(k+2)^2c_0^2+4b_0c_0-$$
$$-2(k-1)(k+3)b_1-2(k+1)(k+3)c_1\Big\}(z-z_0)^2+O((z-z_0)^3)\Big], \tag{19}$$

$$\frac{\varphi'(z)}{\varphi(z)}\frac{F'(z)}{F(z)}=\frac{1}{(z-z_0)}\Big[-(k+3)c_0+\Big\{(k+4)c_0^2+2b_0c_0-2(k+3)c_1\Big\}(z-z_0)+$$
$$+O((z-z_0)^2)\Big]. \tag{20}$$

$$\frac{\varphi'(z)}{\varphi(z)}\frac{h'(z)}{h(z)}=\frac{1}{(z-z_0)}\Big[-(k+3)c_0+\frac{1}{k+3}\Big\{(k^2+7k+11)c_0^2+(k+1)b_0c_0-$$
$$-2(k+3)^2c_1\Big\}(z-z_0)+O((z-z_0)^2)\Big], \tag{21}$$

$$\Big(\frac{\varphi'(z)}{\varphi(z)}\Big)^2=\Big[c_0^2+O((z-z_0))\Big],\qquad\Big(\frac{\varphi'(z)}{\varphi(z)}\Big)'=\Big[(2c_1-c_0^2)+O((z-z_0))\Big]. \tag{22}$$

By substituting the above equalities (15)–(22) in (7) and executing some easy calculation, we obtain $G(z)=O((z-z_0))$. So z_0 is a zero of $G(z)$. □

Lemma 5. *Let $f(z)$, $F(z)$, $\varphi(z)$, $h(z)$ and $G(z)$ be defined as in Lemma 4, $\varphi(z)$ ($\not\equiv 0$) has no zeros of multiplicity 2 and let $k\geq 2$ be an integer. Then $G(z)\not\equiv 0$.*

Proof. We follow the proof as explained in [2, 13, 14]. On the contrary, we assume that $G(z) \equiv 0$. Under this assumption, we first show that: (i) $F(z)$ has no zeros; (ii) $\varphi(z)$ has no zeros and poles; (iii) $h(z)$ has no zeros; (iv) all zeros of $f(z)$ are simple.
Suppose first that z_1 is a zero of $F(z)$ of multiplicity l_1 (≥ 1). From $F(z_1) = 0$ and $F = \varphi f^2 f^{(k)} - 1$ it is obvious that $f(z_1) \neq 0, \infty$ and $\varphi(z_1) \neq 0, \infty$. As z_1 is a zero of $F(z)$ of multiplicity l_1, z_1 is a zero of $F'(z) = fh$ of multiplicity $l_1 - 1$. Obviously, z_1 will be the zero of $h(z)$ of order $l_1 - 1$. Using the Laurent series of $G(z)$ at z_1, we obtain the coefficient of $(z-z_1)^{-2}$ as $A(l_1) = (a_1+a_3+a_4)l_1^2 - (a_2+a_3+2a_4+a_5)l_1 + (a_4+a_5)$. From the definition of a_i ($i \in \{1, 2, , \ldots, 9\}$), we have

$$A(l_1) = -\frac{(k+5)^2(k+7)}{(k+3)} l_1^2 - (k+1)(k+5)(k+7)l_1 + 2(k+1)(k+3)^2.$$

Obviously, $A(l_1) = 0$ has one positive solution lying between 0 and 2. Also $A(1) \neq 0$ for any positive integer k. Hence $A(l_1) \neq 0$ for any positive integers l_1 and k. So the point z_1 is a pole of $G(z)$ which contradicts $G(z) \equiv 0$. Hence our claim (i) is proved. Let z_2 be a zero of $\varphi(z)$ with multiplicity l_2 ($l_2 \geq 1, \neq 2$). Then by (i) and the assumptions of the lemma, $F(z)$ has no zeros and poles at z_2 and also $f(z_2) \neq 0, \infty$. Hence z_2 will be a zero of h of order $l_2 - 1$. Using the Laurent series of $G(z)$ at z_2, we obtain the coefficient of $(z - z_2)^{-2}$ as

$$B(l_2) = (a_4 + a_7 + a_8)l_2^2 - (2a_4 + a_5 + a_7 + a_9)l_2 + (a_4 + a_5).$$

From the definition of a_i we again have

$$B(l_2) = -(k+1)(k+3)l_2^2 - (k+1)^2(k+3)l_2 + 2(k+1)(k+3)^2.$$

Obviously, $B(l_2) \neq 0$ for any positive integers l_2 ($\neq 2$) and k. So the point z_2 is a pole of $G(z)$ which contradicts $G(z) \equiv 0$. Thus if $G(z) \equiv 0$, then $\varphi(z)$ has no zeros.

Now suppose that z_3 is a pole of $\varphi(z)$ of multiplicity l_3 (≥ 1). From $F = \varphi f^2 f^{(k)} - 1$ it is obvious that z_3 will be a pole of $F(z)$ of multiplicity l_3 and a pole of $h(z)$ of multiplicity $l_3 + 1$. Using the Laurent series of $G(z)$ at z_3, we obtain the coefficient of $(z - z_3)^{-2}$ as

$$C(l_3) = (a_1 + a_3 + a_4 + a_6 + a_7 + a_8)l_3^2 + (a_2 + a_3 + 2a_4 + a_5 + a_7 + a_9)l_3 + (a_4 + a_5).$$

From the definition of a_i we have

$$C(l_3) = \frac{2(k+1)(3k+13)}{(k+3)} l_3^2 + 8(k+1)(k+4)l_3 + 2(k+1)(k+3)^2.$$

Obviously, $C(l_3) \neq 0$ for any positive integer l_3. So the point z_3 is a pole of $G(z)$ which contradicts $G(z) \equiv 0$. Thus if $G(z) \equiv 0$, then $\varphi(z)$ has no poles. Hence our claim (ii) is proved.

Now let z_4 be a zero of $h(z)$ of order l_4 (≥ 1). Then by (i), (ii) and the definition of h, $F(z)$ and $\varphi(z)$ has no zeros and poles at z_4. Using the Laurent series of $G(z)$ at z_4, we can get the coefficient of $(z - z_4)^{-2}$ as

$$D(l_4) = a_4 l_4^2 - a_5 l_4.$$

From the definition of a_4 and a_5, we see that $D(l_4) \neq 0$ for any positive integer l_4. So the point z_4 is a pole of $G(z)$ which contradicts $G(z) \equiv 0$. Hence our claim (iii) that $h(z)$ has no zeros is true. From $h(z) = \varphi(z)\{2f'(z)f^{(k)}(z) + f(z)f^{(k+1)}(z)\} + \varphi'(z)f(z)f^{(k)}(z)$ and (iii), we obtain (iv).

Set $\psi(z) = \frac{h(z)}{F(z)}$. We can deduce that $\psi(z)$ is an entire function, all zeros of $\psi(z)$ can occur only at multiple poles of $f(z)$ and the following expressions hold:

$$\frac{F'}{F} = \frac{fh}{F} = f\psi, \qquad \frac{h'}{h} = \frac{F'}{F} + \frac{\psi'}{\psi} = f\psi + \frac{\psi'}{\psi}.$$

Substituting the above two equalities in (7), we get

$$(a_1+a_3+a_4)f^2\psi^2+(a_2+a_3+2a_4+a_5)f\psi'+(a_6+a_7)f\frac{\varphi'}{\varphi}\psi+$$
$$+\Big\{a_4\Big(\frac{\psi'}{\psi}\Big)^2+a_5\Big(\frac{\psi'}{\psi}\Big)'+a_7\frac{\psi'}{\psi}\frac{\varphi'}{\varphi}+a_8\Big(\frac{\varphi'}{\varphi}\Big)^2+a_9\Big(\frac{\varphi'}{\varphi}\Big)'\Big\}+(a_2+a_5)f'\psi\equiv 0. \tag{23}$$

Obviously $a_2+a_5=(k+5)^2(k+7)\neq 0$ and $\psi\not\equiv 0$, otherwise from $\frac{F'}{F}=f\psi\equiv 0$, we get F is equivalently constant which contradicts Lemma 2. Thus by (23), we have

$$f'(z)=\frac{1}{\psi}l_{1,1}(z)+fl_{1,2}(z)+f^2\psi l_{1,3}(z), \tag{24}$$

where $l_{1,i}(z)$ ($i\in\{1,2,3\}$) are differential polynomials in $\frac{\psi'}{\psi}$ and $\frac{\varphi'}{\varphi}$. Differentiating both sides of (24) we have

$$f''(z)=\frac{1}{\psi}l_{2,1}(z)+fl_{2,2}(z)+f^2\psi l_{2,3}(z)+f^3\psi^2 l_{2,4}(z),$$

where $l_{2,i}(z)$ ($i\in\{1,2,3,4\}$) are differential polynomials in $\frac{\psi'}{\psi}$ and $\frac{\varphi'}{\varphi}$. Continuing the above process we obtain

$$f^{(k)}(z)=\frac{1}{\psi}l_{k,1}(z)+fl_{k,2}(z)+f^2\psi l_{k,3}(z)+\ldots+f^{k+1}\psi^k l_{k,k+2}(z), \tag{25}$$

where $l_{k,i}(z)$ ($i\in\{1,2,\ldots,k+2\}$) are differential polynomials in $\frac{\psi'}{\psi}$ and $\frac{\varphi'}{\varphi}$.

Now suppose z_5 is a zero of $f(z)$. From (24) and (25) with $\psi(z_5)\neq 0,\ \infty$, we have

$$f'(z_5)=\frac{1}{\psi(z_5)}l_{1,1}(z_5),\quad f^{(k)}(z_5)=\frac{1}{\psi(z_5)}l_{k,1}(z_5).$$

Also from the expressions for $F(z)$ and $h(z)$ we have

$$F(z_5)=-1,\quad h(z_5)=2\varphi(z_5)f'(z_5)f^{(k)}(z_5)=\frac{2\varphi(z_5)}{\psi^2(z_5)}l_{1,1}(z_5)l_{k,1}(z_5).$$

Substituting the above equality in the expression of $\psi(z)$, we have

$$\psi^3(z_5)=-2\varphi(z_5)l_{1,1}(z_5)l_{k,1}(z_5). \tag{26}$$

Set $\lambda(z)=\psi^3(z)+2\varphi(z)l_{1,1}(z)l_{k,1}(z)$. We now discuss the following two cases: **Case I** and **Case II**.

Case I. Let $\lambda(z)\not\equiv 0$. By (26) and (iv) we have

$$N\Big(r,\frac{1}{f}\Big)=N_1\Big(r,\frac{1}{f}\Big)\le N\Big(r,\frac{1}{\lambda}\Big)<T(r,\lambda)+O(1)<O\{T(r,\psi)\}+S(r,f),$$
$$T(r,\psi)=m(r,\psi)=m\Big(r,\frac{h}{F}\Big)\le m\Big(r,\frac{1}{f}\Big)+S(r,f).$$

Applying (2) and (i), we have

$$m\Big(r,\frac{1}{f}\Big)=S(r,f). \tag{27}$$

Then

$$N\Big(r,\frac{1}{f}\Big)=S(r,f). \tag{28}$$

Combining (27) and (28) we have $T(r,f) = S(r,f)$, a contradiction.
Case II. $\lambda(z) \equiv 0$. Using the expression of $\lambda(z)$, we deduce that

$$T(r,\psi) = m(r,\psi) = S(r,f) + S(r,\psi) = S(r,\psi). \tag{29}$$

Also

$$\psi^3(z) = -2\varphi(z) l_{1,1}(z) l_{k,1}(z). \tag{30}$$

From (29), we deduce that $\psi(z)$ is either a constant or a polynomial. If ψ is a polynomial, then the right-hand side of (30) is either a constant or a rational function whereas the left hand side is a polynomial, and this gives a contradiction. So $\psi(z)(\not\equiv 0)$ is a constant. Let $\psi(z) \equiv C$, where $C(\neq 0)$. Hence from (30) we get

$$\frac{1}{\varphi(z)} = -\frac{2}{C^3} L_{1,1}(z) L_{k,1}(z),$$

where $L_{1,1}(z)$ and $L_{k,1}(z)$ are differential polynomials in $\frac{\varphi'}{\varphi}$. Hence

$$T(r,\varphi) = T\Big(r, \frac{1}{\varphi}\Big) + O(1) = S(r,\varphi).$$

Noting that $\varphi(z)$ has no zero, it becomes a nonzero constant. Substituting $\psi(z) \equiv C$ in (23), we obtain

$$(a_1 + a_3 + a_4) f^2 C^2 + (a_2 + a_5) f' C \equiv 0,$$

which gives $(1/f)' \equiv C_1$, where C_1 $(\neq 0)$ is a constant. Hence $f(z)$ is rational, which is impossible. Thus we get $G(z) \not\equiv 0$. This completes the proof of Lemma 5. □

Lemma 6 ([1]). *Suppose that $f(z)$ is a transcendental meromorphic function and that*

$$f^n P(f) = Q(f),$$

where $P(f)$ and $Q(f)$ are differential polynomials in $f(z)$ with functions of small proximity related to $f(z)$ as the coefficients and the degree of $Q(f)$ is at most n. Then

$$m(r, P(f)) = S(r,f).$$

3. Proof of Theorem 1. By Lemmas 4 and 5, we have seen that the simple poles of $f(z)$ are zeros of $G(z)$ and $G(z) \not\equiv 0$. Differentiating $F = \varphi f^2 f^{(k)} - 1$, we get

$$\beta f = -\frac{F'}{F}, \tag{31}$$

where

$$\beta = \varphi' f f^{(k)} + 2\varphi f' f^{(k)} + \varphi f f^{(k+1)} - \varphi f f^{(k)} \frac{F'}{F}, \qquad h = -\beta F. \tag{32}$$

We see in the proof of Lemma 5 that the poles of $G(z)$ whose multiplicities are at most two come from the multiple poles of $f(z)$ or from the zeros of $F(z)$ or $h(z)$ or from the zeros and poles of $\varphi(z)$ except double zeros of $\varphi(z)$.

We think about the poles of $\beta^2 G$. We can visualize from (32) that the zeros of $h(z)$ are either the zeros of $F(z)$ or the zeros of $\beta(z)$. From (31), we can easily verified that the poles of $f(z)$ with multiplicity q (≥ 2) are the zeros of $\beta(z)$ with multiplicity $q-1$. Hence the poles of $\beta^2 G$ come from the zeros of F(z) and the zeros and poles of $\varphi(z)$ except double zeros of $\varphi(z)$ and the multiplicity is at most 4. Hence,

$$N(r, \beta^2 G) \leq 4\overline{N}(r, 1/F) + 4\overline{N}(r,\varphi) + 4\overline{N}_{\neq 2}(r, 1/\varphi) \leq 4\overline{N}(r, 1/F) + S(r,f).$$

From the expression for $G(z)$ we see that $m(r,G) = S(r,f)$. Also by Lemma 6, we obtain from (31) that $m(r,\beta^2) = S(r,f)$. Therefore $m(r,\beta^2 G) = S(r,f)$. Hence,

$$T(r,\beta^2 G) \le 4\overline{N}(r,1/F) + S(r,f).$$

Since the zeros of $f(z)$ with multiplicity p $(\ge k+1)$ are the zeros of $\beta(z)$ with multiplicity at least $2p-(k+1)$, and therefore at least the zeros of $\beta^2 G$ with multiplicity $4p-2(k+2)$. Also the simple poles $f(z)$ are the zeros of $\beta^2 G$. Hence we have

$$N_1(r,f) + 4N_{(k+1}(r,1/f) - 2(k+2)\overline{N}_{(k+1}(r,1/f) \le N(r,1/\beta^2 G) \le \\ \le T(r,\beta^2 G) \le 4\overline{N}(r,1/F) + S(r,f). \tag{33}$$

Combining twice of (1) and (33), we obtain

$$T(r,f) + N(r,f) + N_1(r,f) - 2\overline{N}(r,f) + 4N\Big(r,\frac{1}{f}\Big) - 2\overline{N}\Big(r,\frac{1}{f}\Big) + 4N_{(k+1}\Big(r,\frac{1}{f}\Big) - \\ -2(k+2)\overline{N}_{(k+1}\Big(r,\frac{1}{f}\Big) - 2N_{k)}\Big(r,\frac{1}{f}\Big) - 2k\overline{N}_{(k+1}\Big(r,\frac{1}{f}\Big) \le \\ \le 6\overline{N}\Big(r,\frac{1}{\varphi f^2 f^{(k)}-1}\Big) - 2N_0\Big(r,\frac{1}{(\varphi f^2 f^{(k)})'}\Big) + S(r,f). \tag{34}$$

Let us consider the poles of left-hand side of (34),

$$N(r,f) + N_1(r,f) - 2\overline{N}(r,f) = N(r,f) + N_1(r,f) - 2N_1(r,f) - 2\overline{N}_{(2}(r,f) = \\ = N(r,f) - N_1(r,f) - 2\overline{N}_{(2}(r,f) = N_1(r,f) + N_{(2}(r,f) - N_1(r,f) - 2\overline{N}_{(2}(r,f) \ge 0. \tag{35}$$

Next we consider the zeros of left-hand side of (34) which is

$$4N\Big(r,\frac{1}{f}\Big) - 2\overline{N}\Big(r,\frac{1}{f}\Big) + 4N_{(k+1}\Big(r,\frac{1}{f}\Big) - 2(k+2)\overline{N}_{(k+1}\Big(r,\frac{1}{f}\Big) - 2N_{k)}\Big(r,\frac{1}{f}\Big) - \\ -2k\overline{N}_{(k+1}\Big(r,\frac{1}{f}\Big) = 2N\Big(r,\frac{1}{f}\Big) - 2\overline{N}\Big(r,\frac{1}{f}\Big) + 4N_{(k+1}\Big(r,\frac{1}{f}\Big) - 2(k+2)\overline{N}_{(k+1}\Big(r,\frac{1}{f}\Big) + \\ +2N_{k)}\Big(r,\frac{1}{f}\Big) + 2N_{(k+1}\Big(r,\frac{1}{f}\Big) - 2N_{k)}\Big(r,\frac{1}{f}\Big) - 2k\overline{N}_{(k+1}\Big(r,\frac{1}{f}\Big) \ge \\ \ge 2N\Big(r,\frac{1}{f}\Big) - 2\overline{N}\Big(r,\frac{1}{f}\Big) + 6N_{(k+1}\Big(r,\frac{1}{f}\Big) - \frac{4k+4}{k+1}N_{(k+1}\Big(r,\frac{1}{f}\Big) > 0. \tag{36}$$

From (34)–(36), we obtain that $T(r,f) \le 6\overline{N}\Big(r,\dfrac{1}{\varphi f^2 f^{(k)}-1}\Big) + S(r,f)$.

4. Open Problems.

We now pose the following two open questions.

Question 4. Is it possible in any way to remove the condition '$\varphi(z)$ has no zero of multiplicity 2' in Theorem 1?

Question 5. What conclusion can be drawn if the set of zeros and poles of $f(z)$ and that of $\varphi(z)$ are not disjoint in Theorem 1?

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