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BOUNDARY VALUE MATRIX PROBLEMS AND DRAZIN INVERTIBLE OPERATORS

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Let A and B be given linear operators on Banach spaces X and Y. We denote by M_C the operator defined on $X \oplus Y$ by $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$. In this paper, we study an abstract boundary value matrix problems with a spectral parameter described by Drazin invertibile operators of the form

$$\begin{cases} U_L = \lambda M_C w + F, \\ \Gamma w = \Phi, \end{cases}$$

where U_L, M_C are upper triangular operators matrices (2×2) acting in Banach spaces, Γ is boundary operator, F and Φ are given vectors and λ is a complex spectral parameter. We introduce the concept of initial boundary operators adapted to the Drazin invertibility and we present a spectral approach for solving the problem. It can be shown that the considered boundary value problems are uniquely solvable and that their solutions are explicitly calculated. As an application we give an example to illustrate our results.

1. Introduction and preliminaries. Many linear boundary value problems in mathematical physics can be written as the abstract equation

$$\begin{cases} U_L = \lambda M_C w + F, \\ \Gamma w = \Phi, \end{cases} \tag{1}$$

where U_L, M_C are upper triangular operators matrices (2×2) acting in Banach spaces, Γ is a boundary operator, F and Φ are given vectors and λ is a complex spectral parameter.

The boundary value problems have been studied by numerous authors, see for example [1, 3, 5, 6, 7, 9] and the references cited therein. This paper is devoted to solving problem (1) in the case where the operator U_L is Drazin invertible.

In order to state our main result, let us introduce some notation and definitions. Let X, Y, E and Z be complex Banach spaces. Let $\mathcal{C}(X, Y)$ and $\mathcal{B}(X, Y)$ denote the set of closed linear operators and bounded linear operators from X into Y, respectively. When $X = Y$, we write $\mathcal{C}(X, X) = \mathcal{C}(X)$ and $\mathcal{B}(X, X) = \mathcal{B}(X)$. The identity operator on a Banach space X is denoted by I_X. The domain of an operator A defined from X into Y is denoted by $\mathcal{D}(A)$, the null space and range of A will be denoted by $\mathcal{N}(A)$ and $\mathcal{R}(A)$, respectively.

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The product AB of two operators A, B defined from X into X is given by

$$BA(x) = B(Ax) \text{ for } x \in \mathcal{D}(BA)$$

where $\mathcal{D}(BA) = \{x \in \mathcal{D}(A)\colon Ax \in \mathcal{D}(B)\}$.

For all $n \in \mathbb{N}$, the domain, the null space and the range of power operator A^n are defined by

$$\mathcal{D}(A^n) := \{x \in \mathcal{D}(A)\colon A^k x \in \mathcal{D}(A),\ k = 1, \dots, n-1\},$$
$$\mathcal{N}(A^n) := \{x \in \mathcal{D}(A^n)\colon A^n x = 0\}, \quad \mathcal{R}(A^n) := \{A^n x\colon x \in \mathcal{D}(A^n)\},$$

and for $n = 0$ by

$$A^0 = I, \quad \mathcal{D}(A^0) = X, \quad \mathcal{N}(A^0) = \{0\}.$$

The ascent $a(A)$ and the descent $d(A)$ of A are given by

$$a(A) = \inf\{n \geq 0\colon \mathcal{N}(A^n) = \mathcal{N}(A^{n+1})\} \text{ and } d(A) = \inf\{n \geq 0\colon \mathcal{R}(A^n) = \mathcal{R}(A^{n+1})\}.$$

An operator $A \in \mathcal{C}(X)$ is said to be *Drazin invertible*, if there exists an operator $S \in \mathcal{B}(X)$ such that

$$SA = AS \quad SAS = S \text{ and } ASA = A + U \text{ where } U \text{ is a nilpotent operator.} \tag{2}$$

The operator S is called a *Drazin inverse* of A, denoted by A^D.

2. Spectral boundary value problems for upper triangular operator matrices. In this section we generalize the results of [7] to the matrix case by studying boundary value problem (1) described by an upper triangular operator matrices (2×2) acting in Banach spaces.

Let U_1 and U_2 be linear operators defined on X and Y, respectively. We denote by U_L the matrix operator defined on $X \oplus Y$ by

$$U_L = \begin{pmatrix} U_1 & L \\ 0 & U_2 \end{pmatrix}$$

for a given linear operator $L\colon Y \longrightarrow X$.

To identify explicitly the unique solution of (1), we first define the initial boundary operators corresponding to Drazin invertible operators and we construct the adapted boundary operator Γ of U_L.

If A^D is the Drazin inverse of the operator A, then

$$\mathcal{D}(A) = \mathcal{R}(A^m) \oplus \mathcal{N}(A^m), \qquad \text{with } \ d(A) = a(A) = m < \infty. \tag{3}$$

Definition 1 ([7])**.** The operator $\Gamma\colon X \to E$ is said to be an *initial boundary operator* for a Drazin invertible operator A corresponding to its Drazin inverse A^D if

(i) $\Gamma A^D = 0$ on $\mathcal{D}(A^D)$;

(ii) there exists an operator $\Pi\colon E \to X$ such that $\Gamma\Pi = I_E$ and $\mathcal{R}(\Pi) = \mathcal{N}(A^m)$ with $m = a(A) = d(A)$.

Lemma 1 ([7], Theorem 3)**.** *Let $A \in \mathcal{C}(X)$ be Drazin invertible operator with $A^D \in \mathcal{B}(X)$. Then there exists an $\varepsilon > 0$ such that $(I - \lambda A^D)$ is invertible for $|\lambda^{-1}| < \varepsilon$ and the boundary value problem* $\begin{cases} (A - \lambda I)x = f, \\ \Gamma x = \varphi \end{cases}$ *has the unique solution*

$$x_\lambda^{f,\phi} = A^D(I - \lambda A^D)^{-1} f + (I - \lambda A^D)^{-1}\Pi\varphi$$

for every $f \in \mathcal{R}(A^m)$, with $a(A) = d(A) = m$.

Theorem 1 ([6])**.** *Let A, B be two linear operators on X such that $\mathcal{R}(A) \subset \mathcal{D}(B)$ and $\mathcal{R}(B) \subset \mathcal{D}(A)$, then $I - \lambda AB$ is invertible if and only if $I - \lambda BA$ is invertible for all $\lambda \neq 0$.*

In this case, we have

$$(I - \lambda BA)^{-1} = I + \lambda B(I - \lambda AB)^{-1}A \tag{4}$$

and

$$(I - \lambda AB)^{-1} = I + \lambda A(I - \lambda BA)^{-1}B. \tag{5}$$

Corollary 1. *Let A, B be two linear operators on X such that $\mathcal{R}(A) \subset \mathcal{D}(B)$ and $\mathcal{R}(B) \subset \mathcal{D}(A)$. If $\lambda^{-1} \in \rho(AB)$ then*

$$(I_X - \lambda AB)^{-1}A = A(I_X - \lambda BA)^{-1}.$$

In the following proposition, we construct the boundary operator for Drazin invertible upper triangular matrix operator.

Proposition 1. *Let $U_L = \begin{pmatrix} U_1 & L \\ 0 & U_2 \end{pmatrix}$ defined on $X \oplus Y$. Assume that U_1^D and U_2^D are Drazin inverses of U_1 and U_2 respectively. Γ_1 and Γ_2 are boundary operators for U_1 and U_2 with the boundary spaces E and Z, respectively. If $\mathcal{N}(U_2^m) \subset \mathcal{N}(L^m)$ with $m = a(U_L) = d(U_L)$ then the operator $\Gamma = \begin{pmatrix} \Gamma_1 & 0 \\ 0 & \Gamma_2 \end{pmatrix}$ from $X \oplus Y$ into $E \oplus Z$ is a boundary operator for U_L.*

Proof. We observe that $\Gamma_1 U_1^D = 0$, $\Gamma_2 U_2^D = 0$ and there exist $\Pi_1 \colon E \to X$ and $\Pi_2 \colon Z \longrightarrow Y$ such that $\Gamma_1\Pi_1 = I_E$, $\quad R(\Pi_1) = \mathcal{N}(U_1^m)$ and $\Gamma_2\Pi_2 = I_Z$, $\quad R(\Pi_2) = \mathcal{N}(U_2^m)$.

Denote by $\Pi = \begin{pmatrix} \Pi_1 & 0 \\ 0 & \Pi_2 \end{pmatrix} : E \oplus Z \longrightarrow X \oplus Y$.

Since U_1 and U_2 are Drazin invertibles, then so is U_L. Let U_L^D be the Drazin inverse of U_L, then $\mathcal{R}(U_L^D) = \mathcal{R}(U_1^D) \oplus \mathcal{R}(U_2^D) \subset \mathcal{N}(\Gamma_1) \oplus \mathcal{N}(\Gamma_2) = \mathcal{N}(\Gamma)$, hence $\Gamma U_L^D = 0$ and $\Gamma\Pi = I_{E\oplus Z}$. The condition $\mathcal{N}(U_2^m) \subset \mathcal{N}(L^m)$ implies that $\mathcal{R}(\Pi) = \mathcal{N}(U_L^m)$. □

Let A and B be given linear operators on Banach spaces X and Y, and consider the operator M_C defined on $X \oplus Y$ by

$$M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix},$$

where C is a linear operator from Y into X such that $\mathcal{D}(U_L) \subset \mathcal{D}(M_C)$.

According to Proposition 1, we define the following spectral boundary value matrix problem for unknown $w \in \mathcal{D}(U_L)$ by

$$(\mathcal{P}) \begin{cases} (U_L - \lambda M_C)w = F, \\ \Gamma w = \Phi, \end{cases}$$

where $F \in X \times Y$, $\Phi \in E \times Z$ and $\lambda \in \mathbb{C}$ is a spectral parameter. We denote $\mathbf{R}_\lambda[U_1^D A] = (I_X - \lambda U_1^D A)^{-1}$ and $\mathbf{R}_\lambda[U_2^D A] = (I_Y - \lambda U_2^D B)^{-1}$, U_1^D and U_2^D are Drazin inverses of U_1 and U_2, respectively.

Our main objective is to establish the existence and uniqueness of solutions for the boundary value problem $(\mathcal{P})$. In the theorem below, we give an explicit expression for the solution of the problem $(\mathcal{P})$.

Theorem 2. *If $\lambda^{-1} \in \rho(U_1^D A) \cap \rho(U_2^D B)$, the boundary value problem $(\mathcal{P})$ is uniquely solvable for any $F \in X \times Y$ and $\Phi \in E \times Z$, the solution is given by*

$$w_\lambda^{F,\Phi} = G_{L,C}(U_L^D F + \Pi\Phi),$$

where $U_L^D = \begin{pmatrix} U_1^D & 0 \\ 0 & U_2^D \end{pmatrix}$ *and* $G_{L,C} = \begin{pmatrix} \mathbf{R}_\lambda[U_1^D A] & -U_1^D \mathbf{R}_\lambda[U_1^D A](L-\lambda C)\mathbf{R}_\lambda[U_2^D B] \\ 0 & \mathbf{R}_\lambda[U_2^D B] \end{pmatrix}$.

Proof. We show that $(U_L - \lambda M_C)w_\lambda^{F,\Phi} = F$, we have

$$(U_L - \lambda M_C)w_\lambda^{F,\Phi} = (U_L - \lambda M_C)G_{L,C}U_L^D F + (U_L - \lambda M_C)G_{L,C}\Pi\Phi.$$

Then

$$\begin{aligned}
&(U_L - \lambda M_C)G_{L,C}U_L^D F = \\
&= (U_L - \lambda M_C)\begin{pmatrix} \mathbf{R}_\lambda[U_1^D A] & -U_1^D \mathbf{R}_\lambda[U_1^D A](L-\lambda C)\mathbf{R}_\lambda[U_2^D B] \\ 0 & \mathbf{R}_\lambda[U_2^D B] \end{pmatrix}\begin{pmatrix} U_1^D f_1 \\ U_2^D f_2 \end{pmatrix} = \\
&= \begin{pmatrix} (U_1 - \lambda A) & (L - \lambda C) \\ 0 & (U_2 - \lambda B) \end{pmatrix} \times \\
&\times \begin{pmatrix} \mathbf{R}_\lambda[U_1^D A]U_1^D f_1 - U_1^D \mathbf{R}_\lambda[U_1^D A](L-\lambda C)\mathbf{R}_\lambda[U_2^D B]U_2^D f_2 \\ \mathbf{R}_\lambda[U_2^D B]U_2^D f_2 \end{pmatrix} = \\
&= \begin{pmatrix} (U_1 - \lambda A)U_1^D \mathbf{R}_\lambda[AU_1^D] f_1 \\ (U_2 - \lambda B)U_2^D \mathbf{R}_\lambda[BU_2^D] f_2 \end{pmatrix} = F,
\end{aligned}$$

and

$$\begin{aligned}
&(U_L - \lambda M_C)G_{L,C}\Pi\Phi = \\
&= \begin{pmatrix} (U_1 - \lambda A)[\mathbf{R}_\lambda[U_1^D A]\Pi_1\varphi_1 - U_1^D \mathbf{R}_\lambda[U_1^D A](L-\lambda C)\mathbf{R}_\lambda[U_2^D B]\Pi_2\varphi_2] \\ +(L-\lambda C)\mathbf{R}_\lambda[U_2^D B]\Pi_2\varphi_2 \\ (U_2 - \lambda B)\mathbf{R}_\lambda[U_2^D B]\Pi_2\varphi_2 \end{pmatrix} = \\
&= \begin{pmatrix} (U_1 - \lambda A)\mathbf{R}_\lambda[U_1^D A]\Pi_1\varphi_1 \\ (U_2 - \lambda B)\mathbf{R}_\lambda[U_2^D B]\Pi_2\varphi_2 \end{pmatrix} = \begin{pmatrix} (U_1 - \lambda A)[I_X + \lambda U_1^D \mathbf{R}_\lambda[AU_1^D]A]\Pi_1\varphi_1 \\ (U_2 - \lambda B)[I_Y + \lambda U_2^D \mathbf{R}_\lambda[BU_2^D]B]\Pi_2\varphi_2 \end{pmatrix} = \\
&= \begin{pmatrix} (U_1 - \lambda A)\Pi_1\varphi_1 + \lambda A\Pi_1\varphi_1 \\ (U_2 - \lambda B)\Pi_2\varphi_2 + \lambda B\Pi_2\varphi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\end{aligned}$$

since $\mathcal{R}(\Pi_1) = \mathcal{N}(U_1^m)$ and $\mathcal{R}(\Pi_2) = \mathcal{N}(U_2^m)$.

Using the fact that $\Gamma_1 U_1^D = 0$ and $\Gamma_2 U_2^D = 0$, we get

$$\begin{aligned}
&\Gamma w_\lambda^{F,\Phi} = \Gamma G_{L,C}(U_L^D F + \Pi\Phi) = \\
&= \begin{pmatrix} \Gamma_1 & 0 \\ 0 & \Gamma_2 \end{pmatrix}\begin{pmatrix} \mathbf{R}_\lambda[U_1^D A]U_1^D f_1 - U_1^D \mathbf{R}_\lambda[U_1^D A](L-\lambda C)\mathbf{R}_\lambda[U_2^D B]U_2^D f_2 \\ \mathbf{R}_\lambda[U_2^D B]U_2^D f_2 \end{pmatrix} + \\
&+ \begin{pmatrix} \Gamma_1 & 0 \\ 0 & \Gamma_2 \end{pmatrix}\begin{pmatrix} \mathbf{R}_\lambda[U_1^D A]\Pi_1\varphi_1 - U_1^D \mathbf{R}_\lambda[U_1^D A](L-\lambda C)\mathbf{R}_\lambda[U_2^D B]\Pi_2\varphi_2 \\ \mathbf{R}_\lambda[U_2^D B]\Pi_2\varphi_2 \end{pmatrix} = \\
&= \begin{pmatrix} \Gamma_1 \mathbf{R}_\lambda[U_1^D A]\Pi_1\varphi_1 - \Gamma_1 U_1^D \mathbf{R}_\lambda[U_1^D A](L-\lambda C)\mathbf{R}_\lambda[U_2^D B]\Pi_2\varphi_2 \\ \Gamma_2 \mathbf{R}_\lambda[U_2^D B]\Pi_2\varphi_2 \end{pmatrix} = \\
&= \begin{pmatrix} \Gamma_1[I_X + \lambda U_1^D \mathbf{R}_\lambda[AU_1^D]A]\Pi_1\varphi_1 \\ \Gamma_2[I_Y + \lambda U_2^D \mathbf{R}_\lambda[BU_2^D]B]\Pi_2\varphi_2 \end{pmatrix} = \begin{pmatrix} \Gamma_1\Pi_1\varphi_1 \\ \Gamma_2\Pi_2\varphi_2 \end{pmatrix} = \Phi.
\end{aligned}$$

If $w_1, w_2 \in \mathcal{D}(U_L)$ are two solutions of $(\mathcal{P})$, then

$$w_0 = w_1 - w_2 = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \begin{pmatrix} U_1^D f_0 + \Pi_1 \varphi_0 \\ U_2^D g_0 + \Pi_2 \psi_0 \end{pmatrix}$$

for $(f_0, g_0) \in X \times Y$, $\varphi_0 \in E$ and $\psi_0 \in Z$. Thus,

$$\begin{cases} (U_L - \lambda M_C) w_0 = 0, \\ \Gamma w_0 = 0. \end{cases}$$

Since $\Gamma_1 U_1^D = 0, \Gamma_2 U_2^D = 0$ and $\Gamma \Pi = I_{E \oplus Z}$, we deduce that $\begin{pmatrix} \varphi_0 \\ \psi_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Then $u_0 = U_1^D f_0$ and $v_0 = U_2^D g_0$. So,

$$0 = (U_L - \lambda M_C) w_0 = \begin{pmatrix} (U_1 - \lambda A) & (L - \lambda C) \\ 0 & (U_2 - \lambda B) \end{pmatrix} \begin{pmatrix} U_1^D f_0 \\ U_2^D g_0 \end{pmatrix} =$$
$$= \begin{pmatrix} (U_1 - \lambda A) U_1^D f_0 + (L - \lambda C) U_2^D g_0 \\ (U_2 - \lambda B) U_2^D g_0 \end{pmatrix}.$$

Then, $f_0 = g_0 = 0$, since $\lambda^{-1} \in \rho(U_1^D A) \cap \rho(U_2^D B)$. Hence, $w_1 = w_2$ and the uniqueness is proved. □

3. Application. We consider the problem

$$\begin{cases} \frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4} = f(x, y), \\ u(0, y) = u_0, \\ u(x, 0) = v_0. \end{cases} \tag{6}$$

We put $v = -i\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$. Then the boundary value problem (6) becomes

$$\frac{\partial^2}{\partial y^2} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} i\frac{\partial^2}{\partial x^2} & 1 \\ 0 & -i\frac{\partial^2}{\partial x^2} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ f \end{pmatrix} \tag{7}$$

with the boundary conditions

$$\begin{cases} u(0, y) = v(0, y) = u_0, \text{ for } y \in \mathbb{R}, \\ u(x, 0) = v(x, 0) = v_0, \text{ for } x \in \mathbb{R}. \end{cases} \tag{8}$$

Let $UCB(I)$ denote the family of all bounded, uniformly continuous complexvalued functions on an interval I and $UCB^k(I)$ be the set of all k times differentiable functions in $UCB(I)$ whose derivatives belong to $UCB(I)$. Let $X = Y = UCB(\mathbb{R})$ be a space equipped with the uniform norm $\|f\| = \sup_{x \in \mathbb{R}} |f(x)|$ and $E = Z = \mathbb{R}$. Consider the operator

$$U = \begin{pmatrix} \frac{\partial^2}{\partial y^2} & 0 \\ 0 & \frac{\partial^2}{\partial y^2} \end{pmatrix}$$

on $X \oplus Y$ with domain $\mathcal{D}(U) = UCB^2(\mathbb{R}) \times UCB^2(\mathbb{R})$.

The null space $\mathcal{N}(U)$ of the operator U is the set of all constant functions on $\mathbb{R}^2$. In [4, Example 3.2] Butzer and Koliha proved that the operator $A = d^2/dx^2$ defined on $X = UCB(\mathbb{R})$ with domain $\mathcal{D}(A) = UCB^2(\mathbb{R})$ is Drazin invertible with $a(A) = d(A) = 1$ and they gave the expression of A^D. According to this example we have U is Drazin invertible with $a(U) = d(U) = 1$ and its Drazin inverse U^D is given by

$$U^D G(x,y) = (I-P)H(x,y) - (QH)(x,y), \quad \text{for} \ \ G \in X \times Y,$$

with

$$U^D = \begin{pmatrix} U_1^D & 0 \\ 0 & U_2^D \end{pmatrix}, \ G = \begin{pmatrix} g_1(x,y) \\ g_2(x,y) \end{pmatrix}, \ PG = \begin{pmatrix} Pg_1 \\ Pg_2 \end{pmatrix}, \ H(x,y) = \begin{pmatrix} h_1(x,y) \\ h_2(x,y) \end{pmatrix}$$

where

$$Pg_k = \lim_{\xi\to\infty} \frac{1}{2\xi} \int_{-\xi}^{\xi} g_k(x,t)dt \text{ for } \xi > 0, \quad h_k(x,y) = \int_0^y \int_0^s (g_k(x,t) - Pg_k(x,t))dtds,$$

$$Qh_k = \lim_{|y|\to\infty} \frac{h_k(x,y)}{y}, \ \text{ for } k = 1, 2.$$

See [4] for more details.

Now, we take

$$A = i\frac{\partial^2}{\partial x^2}, \quad C = I_{UCB(\mathbb{R})}, \quad B = -i\frac{\partial^2}{\partial x^2}$$

and we define the initial boundary operator Γ by

$$\Gamma \left(\begin{array}{c} u(x,y) \\ v(x,y) \end{array} \right) = \left(\begin{array}{c} u_0 \\ v_0 \end{array} \right),$$

with this notations the boundary value problem (6) is equvalent to

$$\begin{pmatrix} Uw = M_C w + F, \\ \Gamma w = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}, \end{pmatrix} \tag{9}$$

where $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$, $w = \begin{pmatrix} u \\ v \end{pmatrix}$ and $F = \begin{pmatrix} 0 \\ f \end{pmatrix} \in \Big(UCB(\mathbb{R})\Big)^2$.

And also define the maps Π by $\Pi \colon \mathbb{R}^2 \to \Big(UCB(\mathbb{R})\Big)^2$ where

$$\left(\Pi \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}\right)(x,y) = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}.$$

Then, $\Gamma\Pi = I_{\mathbb{R}^2}$, $\Gamma\, U^D F = 0$, and $\mathcal{R}(\Pi) = \mathcal{N}(U)$. Due to Lemma 1, the operators $I_X - U_1^D A$ and $I_Y - U_2^D B$ are invertible. According to Theorem 2, the boundary value problem (9) (and hence the problem (6)–(8)) has a unique solution.

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