УДК 517.53

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THE MINIMAL GROWTH OF ENTIRE FUNCTIONS WITH GIVEN ZEROS ALONG UNBOUNDED SETS

I. V. Andrusyak, P. V. Filevych. *The minimal growth of entire functions with given zeros along unbounded sets*, Mat. Stud. **54** (2020), 146–153.

Let l be a continuous function on $\mathbb{R}$ increasing to $+\infty$, and φ be a positive function on $\mathbb{R}$. We proved that the condition

$$\varliminf_{x\to+\infty}\frac{\varphi(\ln[x])}{\ln x}>0$$

is necessary and sufficient in order that for any complex sequence (ζ_n) with $n(r)\ge l(r)$, $r\ge r_0$, and every set $E\subset\mathbb{R}$ which is unbounded from above there exists an entire function f having zeros only at the points ζ_n such that

$$\varliminf_{r\in E,\ r\to+\infty}\frac{\ln\ln M_f(r)}{\varphi(\ln n_\zeta(r))\ln l^{-1}(n_\zeta(r))}=0.$$

Here $n(r)$ is the counting function of (ζ_n), and $M_f(r)$ is the maximum modulus of f.

1. Introduction. Let $\mathcal{Z}$ be the class of all complex sequences

$$\zeta=(\zeta_n) \text{ such that } 0<|\zeta_0|\le|\zeta_1|\le\ldots \text{ and } \zeta_n\to\infty\ (n\to+\infty).$$

For every sequence $\zeta\in\mathcal{Z}$, by $\mathcal{E}_\zeta$ we denote the class of all entire functions whose sequence of zeros, enumerated (counted with multiplicity) in non-decreasing order of their moduli, coincides with the sequence $\zeta\in\mathcal{Z}$, and let

$$n_\zeta(r)=\textstyle\sum_{|\zeta_n|\le r}1$$

be the counting function of this sequence.

Suppose that $E\subset\mathbb{R}$ is a measurable set. As usual, the value $\int_{E\cap(1,+\infty)}d\ln r$ is called the *logarithmic measure* of the set E, and the value

$$\varlimsup_{r\to+\infty}\frac{1}{\ln r}\int_{E\cap(1,+\infty)}d\ln r$$

is called the *upper logarithmic density* of this set.

For an entire function f and every $r\ge0$ we denote $M_f(r)=\max\{|f(z)|:|z|=r\}$.

A. A. Goldberg ([1]) proved the following theorem.

2010 *Mathematics Subject Classification:* 30D15, 30D20.
Keywords: entire function; maximum modulus; zeros; counting function.
doi:10.30970/ms.54.2.146-153

Theorem A. *Let $\delta > 2$. For any sequence $\zeta \in \mathcal{Z}$ satisfying the condition*

$$\varliminf_{r\to+\infty} \frac{\ln n_\zeta(r)}{\ln r} > 0, \tag{1}$$

there exist an entire function $f \in \mathcal{E}_\zeta$ and a set E of finite logarithmic measure such that outside the set E one has

$$\ln\ln M_f(r) = o(\ln^\delta n_\zeta(r)), \quad r \to +\infty. \tag{2}$$

In addition, A. A. Goldberg ([1]) showed that Theorem A is not valid in the case $\delta = 1$, and also posed the question of whether in this theorem the condition $\delta > 2$ can be replaced by the one $\delta > 1$. The negative answer to Goldberg's question was obtained by W. Bergweiler ([2]), who showed that Theorem A is not true anymore even in the case $\delta = 2$.

Theorem B ([2])**.** *Let $\alpha \in (0, +\infty)$. There exists a sequence $\zeta \in \mathcal{Z}$ satisfying the condition*

$$\varliminf_{r\to+\infty} \frac{\ln n_\zeta(r)}{\ln r} = \alpha, \tag{3}$$

such that for any entire function $f \in \mathcal{E}_\zeta$ along some set E_f of infinite logarithmic measure one has

$$\ln^2 n_\zeta(r) = o(\ln\ln M_f(r)), \quad r \to +\infty.$$

If we require the validity of relation (2) not outside a small set such as a set of finite logarithmic measure, but only along some increasing to $+\infty$ sequence of values r, this relation can also be valid in the case of $\delta = 2$.

Theorem C ([2])**.** *For any sequence $\zeta \in \mathcal{Z}$ that satisfies condition* (1) *and every unbounded from above set $E \subset \mathbb{R}$ there exists a function $f \in \mathcal{E}_\zeta$ such that*

$$\varliminf_{r\in E,\ r\to+\infty} \frac{\ln\ln M_f(r)}{\ln^2 n_\zeta(r)} = 0. \tag{4}$$

The following statement shows that under conditions of Theorem C relation (4) is final in some sense.

Theorem D ([2])**.** *Suppose $\alpha \in (0, +\infty)$ and ϕ is a function decreasing to* 0 *on $\mathbb{R}$. Then there exist a sequence $\zeta \in \mathcal{Z}$ satisfying condition* (3) *and a set $E \subset \mathbb{R}$ of upper logarithmic density* 1 *such that for any function $f \in \mathcal{E}_\zeta$ we have*

$$\lim_{r\in E,\ r\to+\infty} \frac{\ln\ln M_f(r)}{\ln^2 n_\zeta(r)\phi(\ln n_\zeta(r))} = +\infty.$$

Remark 1. W. Bergweiler [2] actually proved to some extent deeper results than Theorems A and B. Particularly, from results obtained in [2] it follows that a function $f \in \mathcal{E}_\zeta$ and an exceptional set E in Theorem A can be selected independently of the number $\delta > 2$. On the other hand, Theorem C shows that the set E_f in Theorem B is dependends on f.

Theorems A and B describe the minimal growth of an entire function having given sequence of zeros ζ only in the case when the growth of the counting function $n_\zeta(r)$ for this sequence is bounded from below by some power function r^α. Analogs of Theorems A and B were obtained in [3] in the case when the growth of the function $n_\zeta(r)$ is bounded from below by a function of the form $\ln^\alpha r$. Theorems A and B were extended in [4] to the case when the restrictions mentioned above are of any possible kind.

By L we denote the class of all continuous on $\mathbb{R}$ functions increasing to $+\infty$.

Theorem E ([4])**.** *Let $l \in L$. For any sequence $\zeta \in \mathcal{Z}$ satisfying the condition*

$$n_\zeta(r) \geq l(r) \quad (r \geq r_0), \tag{5}$$

there exist an entire function $f \in \mathcal{E}_\zeta$ and a set $E \subset \mathbb{R}$ of finite logarithmic measure such that for every $\delta > 1$ outside the set E one has

$$\ln\ln M_f(r) = o(\ln^\delta n_\zeta(r) \ln l^{-1}(n_\zeta(r))), \quad r \to +\infty.$$

Theorem F ([4])**.** *Let $l \in L$. There exists a sequence $\zeta \in \mathcal{Z}$ that satisfies condition* (5) *such that $n_\zeta(r-0) = l(r)$ on an unbounded from above set of values r and for any entire function $f \in \mathcal{E}_\zeta$ along some set E_f of infinite logarithmic measure one has*

$$\ln n_\zeta(r) \ln l^{-1}(n_\zeta(r)) = o(\ln\ln M_f(r)), \quad r \to +\infty.$$

The goal of our paper is generalizations of Theorems C and D for the case of any possible lower bound on the growth of the counting function $n_\zeta(r)$ for a sequence $\zeta \in \mathcal{Z}$.

Theorem 1. *Let $l \in L$. Then for any sequence $\zeta \in \mathcal{Z}$ that satisfies condition* (5)*, and for every unbounded from above set $E \subset \mathbb{R}$ there exists a function $f \in \mathcal{E}_\zeta$ for which*

$$\varliminf_{r\in E,\ r\to+\infty} \frac{\ln\ln M_f(r)}{\ln n_\zeta(r) \ln l^{-1}(n_\zeta(r))} = 0. \tag{6}$$

Theorem 2. *Let $l \in L$ and φ be a function which is positive on $\mathbb{R}$ and such that*

$$\varliminf_{x\to+\infty} \frac{\varphi(\ln[x])}{\ln x} = 0. \tag{7}$$

Then there exists a sequence $\zeta \in \mathcal{Z}$ such that condition (5) *holds and $n_\zeta(r-0) = l(r)$ on an unbounded from above set of values r, and also there exists a set $E \subset \mathbb{R}$ of upper logarithmic density* 1 *such that for any function $f \in \mathcal{E}_\zeta$ one has*

$$\lim_{r\in E,\ r\to+\infty} \frac{\ln\ln M_f(r)}{\varphi(\ln n_\zeta(r)) \ln l^{-1}(n_\zeta(r))} = +\infty. \tag{8}$$

Immediately from Theorems 1 and 2 we obtain the following theorem.

Theorem 3. *Let $l \in L$ and φ be a function which is positive on $\mathbb{R}$. Then the condition*

$$\varliminf_{x\to+\infty} \frac{\varphi(\ln[x])}{\ln x} > 0$$

is necessary and sufficient in order that for any sequence $\zeta \in \mathcal{Z}$ that satisfies condition (5) *and for every unbounded from above set $E \subset \mathbb{R}$ there exists an entire function $f \in \mathcal{E}_\zeta$ such that*

$$\varliminf_{r\in E,\ r\to+\infty} \frac{\ln\ln M_f(r)}{\varphi(\ln n_\zeta(r)) \ln l^{-1}(n_\zeta(r))} = 0.$$

To prove Theorems 1 and 2, we need some auxiliary results which are formulated in the next section.

We have noted that the paper [1] also inspired other problems related to the description of the minimum growth of entire functions with given zeros. In particular, some of these problems were solved in papers [5, 6, 7, 8, 9].

2. Auxiliary results.

Lemma 1 ([4]). *For every sequence $\zeta \in \mathcal{Z}$ there exists a nonnegative sequence (λ_n) having the following properties:*

(i) $\lambda_n \sim \ln n / \ln |\zeta_n|$, *as* $n \to \infty$;

(ii) for any sequence of nonnegative integers (p_n) such that $p_n \geq [\lambda_n]$, $n \geq n_0$, the product

$$f(z) = \prod_{n=0}^{\infty} E\left(\frac{z}{\zeta_n}, p_n\right) \tag{9}$$

defines an entire function $f \in \mathcal{E}(\zeta)$, moreover,

$$\ln M_f(r) \leq G_f(r) := \sum_{n=0}^{\infty} \left(\frac{r}{|\zeta_n|}\right)^{p_n+1}. \tag{10}$$

Let f be an entire function, $r > 0$, and $c_p(r)$ be the p-th Fourier coefficient of the function $\ln |f(re^{i\theta})|$, that is,

$$c_p(r) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ip\theta} \ln |f(re^{i\theta})| d\theta, \quad p \in \mathbb{Z}.$$

Suppose that $f(0) \neq 0$ and

$$\ln f(z) = \sum_{p=0}^{\infty} a_p z^p \tag{11}$$

near the point $z = 0$. Then, applying the Poisson-Jensen formula (see [10, p. 16–17]), for every integer $p \geq 1$ we have

$$c_p(r) = \frac{1}{2} a_p r^p + \frac{1}{2p} \sum_{|\zeta_n| < r} \left(\left(\frac{r}{\zeta_n}\right)^p - \left(\frac{\overline{\zeta_n}}{r}\right)^p \right), \tag{12}$$

where ζ_n are zeros of the function f. Moreover, the following lemma is valid.

Lemma 2 ([2]). *For any entire function f and every integer $n \geq 1$ the inequality*

$$|c_n(r)| \leq \ln M_f(r) \quad (r > 0)$$

is satisfied.

3. Proof of Theorems.

Proof of Theorem 1. Suppose that $\zeta \in \mathcal{Z}$ is a fixed sequence such that $n_\zeta(r) \ge l(r)$, $r \ge r_0$, and $\lambda = (\lambda_n)$ is a sequence whose existence for given ζ is asserted by Lemma 1. Then there exists a non-decreasing sequence of non-negative integers (q_n) such that $q_n \ge [\lambda_n]$ for $n \ge n_0$, and also $q_n \to +\infty$ and $q_n = o(\ln n)$ as $n \to \infty$.

Let us consider the series

$$\sum_{n=0}^{\infty} \left(\frac{r}{|\zeta_n|}\right)^{q_n+1},$$

which is convergent for all $r \ge 0$, and for every $r \ge 0$ we put $n(r) = n_\zeta(r)$. We also put

$$m(r) = \min\left\{k \ge n(r) + 2 : \sum_{n=k}^{\infty} \left(\frac{r}{|\zeta_n|}\right)^{q_n+1} \le 1\right\}, \quad \gamma(r) = \frac{\ln(m(r) - n(r))}{\ln|\zeta_{n(r)+1}| - \ln r}.$$

Let $E \subset \mathbb{R}$ be a set unbounded from above. In this set we choose a sequence (r_k) increasing to $+\infty$ such that for every integer $k \ge 0$ the following inequality is valid:

$$n(r_{k+1}) > m(r_k), \quad q_{n(r_{k+1})} \ge \gamma(r_k) + 1.$$

For any integer $t \ge 0$ let us denote $p_t = \max\{[\gamma(r_k)] + 1, q_t\}$ if $t \in [n(r_k) + 1, m(r_k))$ for some $k \ge 0$, and put $p_t = q_t$ if $t \notin H$, where

$$H = \bigcup_{k=0}^{\infty} [n(r_k) + 1, m(r_k)).$$

Note that $n(r_k) \notin H$ for every $k \ge 0$.

Let us prove that $p_t \le q_{n(r_k)}$ for all $t \le n(r_k)$ and $k \ge 0$. If $t \notin H$, we have $p_t = q_t \le q_{n(r_k)}$, because the sequence (q_n) is non-decreasing. But if $t \in H$, then we have $t \in [n(r_j)+1, m(r_j))$ for some $j < k$, and therefore either

$$p_t = [\gamma(r_j)] + 1 \le \gamma(r_j) + 1 \le q_{n(r_{j+1})} \le q_{n(r_k)},$$

or $p_t = q_t \le q_{n(r_k)}$ again.

It is also clear that $p_t \ge q_t$ for all $t \ge 0$. Therefore, by Lemma 1, product (9) defines an entire function $f \in \mathcal{E}_\zeta$ for which inequality (10) is true. For each $k \ge 0$, we successively have

$$\sum_{t \le n(r_k)} \left(\frac{r_k}{|\zeta_t|}\right)^{p_t+1} \le r_k^{q_{n(r_k)}+1} \sum_{t \le n(r_k)} \left(\frac{1}{|\zeta_t|}\right)^{p_t+1} \le r_k^{q_{n(r_k)}+1} G(1),$$

$$\sum_{n(r_k) < t < m(r_k)} \left(\frac{r_k}{|\zeta_t|}\right)^{p_t+1} \le (m(r_k) - n(r_k)) \left(\frac{r_k}{|\zeta_{n(r_k)+1}|}\right)^{\gamma(r_k)} = 1,$$

$$\sum_{t \ge m(r_k)} \left(\frac{r_k}{|\zeta_t|}\right)^{p_t+1} \le \sum_{t \ge m(r_k)} \left(\frac{r_k}{|\zeta_t|}\right)^{q_t+1} \le 1.$$

Thus, applying inequality (10), we obtain

$$\ln\ln M_f(r_k) \le (1 + o(1)) q_{n(r_k)} \ln r_k = o(\ln n(r_k) \ln l^{-1}(n(r_k))), \quad k \to \infty.$$

It implies (6). □

Proof of Theorem 2. Without loss of generality, we may suppose that $l(1) < 0$.

Let (δ_k) be any decreasing to 0 sequence of points in the interval $(0,1)$. From condition (7) it follows the existence of an increasing sequence of integers (n_k) such that $n_0 = 0$ and

$$2r_k \le r_{k+1}^{\delta_k}, \quad \ln m_k \ge (1-\delta_k)\ln n_{k+1}, \tag{13}$$

$$\ln m_k \ge 2k \ln r_k, \quad \varepsilon_k \le \delta_k, \quad \varepsilon_k \ln r_k \le 1, \tag{14}$$

for every $k \ge 0$, where

$$\varepsilon_k = \left(\frac{\varphi(\ln n_{k+1})}{\ln n_{k+1}}\right)^{1/3}, \quad r_k = l^{-1}(n_k), \quad m_k = n_{k+1} - n_k \quad (k \ge 0).$$

Note that $r_0 > 1$ by inequality $l(1) < 0$.

For any $k \ge 0$ we also denote

$$p_k = \left[\frac{\ln m_k}{2\ln r_k}\right] + 1.$$

Applying the first inequality in (14), we see that $p_k \to +\infty$, $k \to \infty$. Moreover,

$$\ln \frac{m_k}{4p_k r_k^{p_k}} = \ln m_k - p_k \ln r_k - \ln p_k - \ln 4 = \left(\frac{1}{2} + o(1)\right)\ln m_k \to +\infty, \quad k \to \infty. \tag{15}$$

Construct the sequence ζ as following

$$\underbrace{r_0,\dots,r_0}_{m_0}, \underbrace{r_1,\dots,r_1}_{m_1}, \dots, \underbrace{r_k,\dots,r_k}_{m_k}, \dots.$$

If $r \in [0, r_0)$, then $n_\zeta(r) = 0 = l(r_0) > l(r)$. But if $r \in [r_k, r_{k+1})$ for some $k \ge 0$, then we get

$$n_\zeta(r) = \sum_{j=0}^{k} m_j = n_{k+1} = l(r_{k+1}) > l(r).$$

Therefore, $n_\zeta(r) > l(r)$ for every $r \ge 0$. Moreover, $n_\zeta(r_k - 0) = l(r_k)$ for all $k \ge 0$.

Denote $s_k = r_{k+1}^{\delta_k}$, $k \ge 0$. Then, applying the second inequality in (13), we see that $r_k < s_k < r_{k+1}$ for every $k \ge 0$. Let $E = \cup_{k=0}^{\infty}(s_k, r_{k+1})$. For the set E we have

$$\overline{\lim_{r\to+\infty}} \frac{1}{\ln r}\int_{E\cap(1,r)} \frac{dt}{t} \ge \overline{\lim_{k\to\infty}} \frac{1}{\ln r_{k+1}}\int_{s_k}^{r_{k+1}} \frac{dt}{t} = \overline{\lim_{k\to\infty}}(1-\delta_k) = 1.$$

Thus, E is a set of upper logarithmic density 1.

We need to prove that for any function $f \in \mathcal{E}_\zeta$ relation (8) is true.

Let $f \in \mathcal{E}_\zeta$. Then in the disc $\{z \in \mathbb{C} : |z| < r_0\}$ the function f has not zeros. Since $r_0 > 1$, we deduce that if we have (11) near the point $z = 0$, then the sequence (a_p) is bounded, that is, $C := \sup\{|a_p| : p \in \mathbb{N}\} < +\infty$.

Suppose $c_p(r)$ is the p-th Fourier coefficient of the function $\ln|f(re^{i\theta})|$. Using equality (12) and the first inequality in (13), which can be rewritten as $2r_k \le s_k$, and, taking into account that the sequence ζ is positive for all $k \geq 0$ and $p \geq 1$, we obtain

$$|c_p(s_k)| \geq \frac{1}{2p} \sum_{\zeta_n < s_k} \left(\left(\frac{s_k}{\zeta_n} \right)^p - \left(\frac{\zeta_n}{s_k} \right)^p \right) - C s_k^p \geq \frac{1}{2p} \sum_{r_k \leq \zeta_n < s_k} \left(\left(\frac{s_k}{\zeta_n} \right)^p - \left(\frac{\zeta_n}{s_k} \right)^p \right) - C s_k^p =$$

$$= \frac{m_k}{2p} \left(\left(\frac{s_k}{r_k} \right)^p - \left(\frac{r_k}{s_k} \right)^p \right) - C s_k^p = \frac{m_k}{2p} \left(\frac{s_k}{r_k} \right)^p \left(1 - \left(\frac{r_k}{s_k} \right)^{2p} \right) - C s_k^p \geq$$

$$\geq \frac{m_k}{4p} \left(\frac{s_k}{r_k} \right)^p - C s_k^p \geq s_k^p \left(\frac{m_k}{4p r_k^p} - C \right).$$

Further, from (15), it follows that $|c_{p_k}(s_k)| \geq s_k^{p_k}$, $k \geq k_0$.

Using this inequality together with (13) and (14), for all $r \in (s_k, r_{k+1})$ and $k \geq k_0$ we obtain

$$\ln \ln M_f(r) \geq \ln \ln M_f(s_k) \geq \ln |c_p(s_k)| \geq p_k \ln s_k \geq$$

$$\geq \frac{\ln m_k}{2 \ln r_k} \delta_k \ln r_{k+1} \geq \frac{1-\delta_k}{2} \ln n_{k+1} \frac{1}{\ln r_k} \delta_k \ln l^{-1}(n_{k+1}) \geq$$

$$\geq \frac{1-\delta_k}{2} \frac{\varphi(\ln n_{k+1})}{\varepsilon_k^3} \varepsilon_k \varepsilon_k \ln l^{-1}(n_{k+1}) = \frac{1-\delta_k}{2\varepsilon_k} \varphi(\ln n_\zeta(r)) \ln l^{-1}(n_\zeta(r)).$$

Hence, we get (8). □

Remark 2. To characterize the growth of an entire function f, we can use besides $\ln M_f(r)$ its Nevanlinna characteristic function

$$T_f(r) = \frac{1}{2\pi} \int_0^{2\pi} \ln^+ |f(re^{i\theta})| d\theta, \quad r \geq 0.$$

We note that in Theorems 1, 2, and 3 the function $\ln M_f(r)$ can be replaced by the function $T_f(r)$. The validity of this replacement in relation (6) follows from the inequality $T_f(r) \leq \ln^+ M_f(r)$. To justify the possibility of replacing $\ln M_f(r)$ by $T_f(r)$ in relation (8), it is sufficient to repeat the proof of Theorem 2 using the inequality $|c_n(r)| \leq 2T_f(r)$ (see, for example, [10, p. 340]) instead of the inequality $|c_n(r)| \leq \ln M_f(r)$ from Lemma 2.

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Received 30.06.2020
Revised 24.10.2020