УДК 517.537

M. M. Sheremeta

HADAMARD COMPOSITIONS OF GELFOND-LEONT'EV-SĂLĂGEAN AND GELFOND-LEONT'EV-RUSCHEWEYH DERIVATIVES OF FUNCTIONS ANALYTIC IN THE UNIT DISK

M. M. Sheremeta. Hadamard compositions of Gelfond-Leont'ev-Sălăgean and Gelfond-Leont'ev-Ruscheweyh derivatives of functions analytic in the unit disk, Mat. Stud. **54** (2020), 115–134.

For analytic functions

$$f(z)=z+\sum_{k=2}^{\infty}f_kz^k \text{ and } g(z)=z+\sum_{k=2}^{\infty}g_kz^k$$

in the unit disk properties of the Hadamard compositions $D^n_{l,[S]}f*D^n_{l,[S]}g$ and $D^n_{l,[R]}f*D^n_{l,[R]}g$ of their Gelfond-Leont'ev-Sălăgean derivatives

$$D^n_{l,[S]}f(z)=z+\sum_{k=2}^{\infty}\left(\frac{l_1l_{k-1}}{l_k}\right)^n f_kz^k$$

and Gelfond-Leont'ev-Ruscheweyh derivatives

$$D^n_{l,[R]}f(z)=z+\sum_{k=2}^{\infty}\frac{l_{k-1}l_n}{l_{n+k-1}}f_kz^k$$

are investigated. For study, generalized orders are used. A connection between the growth of the maximal term of the Hadamard composition of Gelfond-Leont'ev-Sălăgean derivatives or Gelfond-Leont'ev-Ruscheweyh derivatives and the growth of the maximal term of these derivatives of Hadamard composition is established. Similar results are obtained in terms of the classical order and the lower order of the growth.

1. Introduction. For formal power series

$$f(z)=\sum_{k=0}^{\infty}f_kz^k \text{ and } l(z)=\sum_{k=0}^{\infty}l_kz^k$$

($l_k>0$) the formal power series

$$D^n_lf(z)=\sum_{k=0}^{\infty}\frac{l_k}{l_{k+n}}f_{k+n}z^k$$

2010 *Mathematics Subject Classification:* 30B10, 30B40.
Keywords: analytic function; Gelfond-Leont'ev-Sălăgean derivative; Gelfond-Leont'ev-Ruscheweyh derivative; Hadamard composition; generalized order.
doi:10.30970/ms.54.2.115-134

is called the *Gelfond-Leont'ev derivative* ([1]). If $l(z)=e^z$ (i.e. $l_k=1/k!$) then $D_l^n f=f^{(n)}$ is a usual derivative.

Let H be the class of analytic in the disk $\{z:|z|<1\}$ functions given by power series

$$f(z)=z+\sum_{k=2}^{\infty} f_k z^k \tag{1}$$

with the radius of convergence $R[f]=1$ and the operator $D_{[S]}^n f (n\ge 0)$ be defined by $D_{[S]}^0 f(z)=f(z)$, $D_{[S]}^1 f(z)=D_{[S]}f(z)=zf'(z)$ and

$$D_{[S]}^n f(z)=D_{[S]}(D_{[S]}^{n-1}f(z))=z+\sum_{k=2}^{\infty} k^n f_k z^k.$$

The operator $D_{[S]}^n f$ is known as the *Sălăgean derivative* ([2]). For $f\in H$

$$D_{[R]}^n f(z)=\frac{z}{n!}\frac{d^n}{dz^n}\{z^{n-1}f(z)\}=z+\sum_{k=2}^{\infty}\frac{(k+n-1)!}{n!(k-1)!}f_k z^k$$

is called the *Ruscheweyh derivative* ([3]).

Combining the definitions of Gelfond-Leont'ev derivative with Sălăgean derivative and Ruscheweyh derivative in [4] for $f\in H$ the following operators are defined

$$D_{l,[S]}^n f(z)=l_1 z D_l^1(D_{l,[S]}^{n-1}f(z))=z+\sum_{k=2}^{\infty}\left(\frac{l_1 l_{k-1}}{l_k}\right)^n f_k z^k \tag{2}$$

and

$$D_{l,[R]}^n f(z)=z l_n D_l^n\{z^{n-1}f(z)\}=z+\sum_{k=2}^{\infty}\frac{l_{k-1}l_n}{l_{n+k-1}}f_k z^k. \tag{3}$$

The operator $D_{l,[S]}^n$ is called the *Gelfond-Leont'ev-Sălăgean derivative* ([4]) and the operator $D_{l,[R]}^n$ is called the Gelfond-Leont'ev-Ruscheweyh derivative.

For power series (1) and $g(z)=\sum_{k=0}^{\infty} g_k z^k$ with the convergence radii $R[f]$ and $R[g]$ the series

$$(f*g)(z)=\sum_{k=0}^{\infty} f_k g_k z^k$$

is called [5, 6] the *Hadamard composition.* Obtained by J. Hadamard properties of this composition find the applications ([6, 7]) in the theory of the analytic continuation of the functions represented by power series. We remark also that singular points of the Hadamard composition are investigated in the paper [8].

For $0\le r<R[f]$ let $M(r,f)=\max\{|f(z)|:|z|=r\}$ and $\mu(r,f)=\max\{|f_k|r^k:k\ge 0\}$ be the maximal term of the power expansion of f and $\nu(r,f)=\max\{n:|f_n r^n=\mu(r,f)\}$ be its central index. A connection between the growth of the maximal terms of a derivative of the Hadamard composition of two entire functions f and g and the Hadamard composition of their derivative is studied by M. K. Sen ([9, 10]). The properties of compositions of Hadamard for Gelfond-Leont'ev derivatives of analytic functions f and g are investigated in [11]. For entire functions f and g it is proved ([11]) for example that if

$$0<\varliminf_{k\to\infty}\frac{l_k}{(k+1)l_{k+1}}\le\varlimsup_{k\to\infty}\frac{l_k}{(k+1)l_{k+1}}<+\infty$$

then

$$\varlimsup_{r\to+\infty}\frac{1}{\ln r}\ln\frac{\mu(r,D_l^{(n+1)}f*D_l^{(n+1)}g)}{\mu(r,D_l^{(n)}(f*g))}=(n+2)\varrho[f*g]-1$$

and

$$\varliminf_{r\to+\infty}\frac{1}{\ln r}\ln\frac{\mu(r,D_l^{(n+1)}f*D_l^{(n+1)}g)}{\mu(r,D_l^{(n)}(f*g))}=(n+2)\lambda[f*g]-1,$$

where $\varrho[f]$ is the order and $\lambda[f]$ is the lower order of the entire function f. If $R[f]=1$, $R[g]=1$ and $R[f*g]=1$ then ([11])

$$(n+2)\varrho^{(1)}[f*g]\le\varlimsup_{r\uparrow 1}\frac{1}{-\ln(1-r)}\ln^+\frac{\mu(r,D_l^{(n+1)}f*D_l^{(n+1)}g)}{\mu(r,D_l^{(n)}(f*g))}\le(n+2)(\varrho^{(1)}[f*g]+1)$$

and

$$(n+2)\lambda^{(1)}[f*g]\le\varliminf_{r\uparrow 1}\frac{1}{-\ln(1-r)}\ln^+\frac{\mu(r,D_l^{(n+1)}f*D_l^{(n+1)}g)}{\mu(r,D_l^{(n)}(f*g))}\le(n+2)(\lambda^{(1)}[f*g]+1),$$

where $\varrho^{(1)}[f]$ is the order and $\lambda^{(1)}[f]$ is the lower order of the analytic function f in the unit disk.

Naturally, the question arises of similar properties of the Hadamard compositions of the Gelfond-Leont'ev-Sălăgean derivatives and the Gelfond-Leont'ev-Ruscheweyh derivatives.

2. Hadamard composition of two Gelfond-Leont'ev-Sălăgean derivatives. Let $f\in H$, $g\in H$, $n\in\mathbb{Z}_+$ and $m\in\mathbb{Z}_+$. Then

$$(D^n_{l,[S]}f*D^m_{l,[S]}g)(z)=z+\sum_{k=2}^{\infty}\left(\frac{l_1l_{k-1}}{l_k}\right)^n f_k\left(\frac{l_1l_{k-1}}{l_k}\right)^m g_kz^k=$$

$$=z+\sum_{k=2}^{\infty}\left(\frac{l_1l_{k-1}}{l_k}\right)^{n+m}f_kg_kz^k=D^{n+m}_{l,[S]}(f*g)(z),\tag{4}$$

i. e. the study of the Hadamard composition of Gelfond-Leont'ev-Sălăgean derivatives of two functions is reduced to the study of Gelfond-Leont'ev-Sălăgeanthe derivative of the Hadamard composition of these functions. For the Gelfond-Leont'ev-Sălăgeanthe derivative of the Hadamard composition the following statement is true.

Lemma 1. *If $f\in H$, $g\in H$, $n\in\mathbb{N}$ and there exists $\lim\limits_{k\to\infty}\sqrt[k]{l_{k-1}/l_k}=q$ then $R[D^n_{l,[S]}(f*g)]\ge q^{-n}$ and if, moreover, $\lim\limits_{k\to\infty}\sqrt[k]{|g_k|}=1$ then $R[D^n_{l,[S]}(f*g)]=q^{-n}$.*

Proof. Indeed,

$$\frac{1}{R[D^n_{l,[S]}(f*g)]}=\varlimsup_{k\to\infty}\sqrt[k]{\left(\frac{l_1l_{k-1}}{l_k}\right)^n|f_k||g_k|}\le$$

$$\le\left(\varlimsup_{k\to\infty}\sqrt[k]{\frac{l_{k-1}}{l_k}}\right)^n\varlimsup_{k\to\infty}\sqrt[k]{|f_k|}\,\varlimsup_{k\to\infty}\sqrt[k]{|g_k|}=\left(\varlimsup_{k\to\infty}\sqrt[k]{\frac{l_{k-1}}{l_k}}\right)^n,$$

$$\frac{1}{R[D^n_{l,[S]}(f*g)]}\ge\left(\varliminf_{k\to\infty}\sqrt[k]{\frac{l_{k-1}}{l_k}}\right)^n\varlimsup_{k\to\infty}\sqrt[k]{|f_k|}\,\varliminf_{k\to\infty}\sqrt[k]{|g_k|}.$$

Therefore, if there exists $\lim\limits_{k\to\infty}\sqrt[k]{l_{k-1}/l_k}=q$ then $1/R[D^n_{l,[S]}(f*g)]\le q^n$, and if, moreover, $\lim\limits_{k\to\infty}\sqrt[k]{|g_k|}=1$ then $1/R[D^n_{l,[S]}(f*g)]\ge q^n$. Lemma 1 is proved. □

Let E be the class of entire functions. From Lemma 1 it follows that if $\sqrt[k]{l_k/l_{k-1}}\to\infty$ as $k\to\infty$ then $D^n_{l,[S]}(f*g)\in E$. If $\sqrt[k]{l_k/l_{k-1}}\to 1$ and $\lim\limits_{k\to\infty}\sqrt[k]{|g_k|}=1$ as $k\to\infty$ then $D^n_{l,[S]}(f*g)\in H$. In the sequel, we will consider only these two cases.

To study the growth of analytic functions, we will use generalized orders. For this purpose by L we denote the class of continuous non-negative on $(-\infty,+\infty)$ functions α such that $\alpha(x)=\alpha(x_0)>0$ for $x\le x_0$ and $\alpha(x)\uparrow+\infty$ as $x_0\le x\to+\infty$. We say that $\alpha\in L^0$, if $\alpha\in L$ and $\alpha((1+o(1))x)=(1+o(1))\alpha(x)$ as $x\to+\infty$. Finally, $\alpha\in L_{si}$, if $\alpha\in L$ and $\alpha(cx)=(1+o(1))\alpha(x)$ as $x\to+\infty$ for each fixed $c\in(0,+\infty)$, i. e. α is a slowly increasing function. Clearly, $L_{si}\subset L^0$.

2.1. The case $\lim\limits_{k\to\infty}\sqrt[k]{l_k/l_{k-1}}=+\infty$. For $\alpha\in L$, $\beta\in L$ and an entire transcendental function (1) the quantities

$$\varrho_{\alpha,\beta}[f]:=\varrho_{\alpha,\beta}[\ln M,f]=\varlimsup_{r\to+\infty}\frac{\alpha(\ln M(r,f))}{\beta(\ln r)},$$
$$\lambda_{\alpha,\beta}[f]:=\lambda_{\alpha,\beta}[\ln M,f]=\varliminf_{r\to+\infty}\frac{\alpha(\ln M(r,f))}{\beta(\ln r)}$$

are called ([12]) the *generalized order and the lower generalized order*, respectively. If we substitute $\ln\mu(r,f)$ or $\nu(r,f)$ instead of $\ln M(r,f)$ then we obtain the definitions of the quantities $\varrho_{\alpha,\beta}[\ln\mu,f]$, $\lambda_{\alpha,\beta}[\ln\mu,f]$ and $\lambda_{\alpha,\beta}[\nu,f]$, $\lambda_{\alpha,\beta}[\nu,f]$, respectively.

Lemma 2. *Let $\alpha\in L_{si}$, $\beta\in L^0$ and $\frac{d\beta^{-1}(c\alpha(x))}{d\ln x}=O(1)$ as $x\to+\infty$ for each $c\in(0,+\infty)$. Then*

$$\varrho_{\alpha,\beta}[f]=\varlimsup_{k\to\infty}\alpha(k)/\beta\left(\frac{1}{k}\ln\frac{1}{|f_k|}\right). \tag{5}$$

If, moreover, $|f_k/f_{k+1}|\nearrow+\infty$ as $k_0\le k\to\infty$ then

$$\lambda_{\alpha,\beta}[f]=\varliminf_{k\to\infty}\alpha(k)/\beta\left(\frac{1}{k}\ln\frac{1}{|f_k|}\right). \tag{6}$$

Formula (5) is proved in [12], and formula (6) follows from the corresponding formula for entire Dirichlet series proved in [13].

The following lemma is proved in [14].

Lemma 3. *If $\alpha\in L_{si}$ and $\beta\in L^0$ then $\varrho_{\alpha,\beta}[f]=\varrho_{\alpha,\beta}[\ln\mu,f]$ and $\lambda_{\alpha,\beta}[f]=\lambda_{\alpha,\beta}[\ln\mu,f]$. If, moreover, $\alpha(e^x)\in L_{si}$ and $\alpha(x)=o(\beta(x))$ as $x\to+\infty$ then $\varrho_{\alpha,\beta}[\ln\mu,f]=\varrho_{\alpha,\beta}[\nu,f]$ and $\lambda_{\alpha,\beta}[\ln\mu,f]=\lambda_{\alpha,\beta}[\nu,f]$.*

Theorem 1. *Let $\alpha(x)=\alpha_1(\ln x)$, $\alpha_1\in L_{si}$, $\beta\in L_{si}$ and $\frac{d\beta^{-1}(c\alpha(x))}{d\ln x}=O(1)$ as $x\to+\infty$ for each $c\in(0,+\infty)$. Suppose that $\varliminf\limits_{k\to\infty}\sqrt[k]{|f_kg_k|}>0$ and*

$$0<q=\varliminf_{k\to\infty}\frac{1}{\ln k}\ln\ln\frac{l_k}{l_{k-1}}\le\varlimsup_{k\to\infty}\frac{1}{\ln k}\ln\ln\frac{l_k}{l_{k-1}}=Q<+\infty. \tag{7}$$

Then for $m > n \geq 1$

$$\varlimsup_{r\to+\infty} \frac{1}{\beta(\ln r)}\alpha\left(\ln \frac{\mu(r, D^n_{l,[S]}(f*g))}{\mu(r, D^m_{l,[S]}(f*g))}\right) = \varlimsup_{k\to\infty} \frac{\alpha(k)}{\beta(\frac{1}{k}\ln\frac{l_k}{l_{k-1}})} \tag{8}$$

and if, moreover, $|f_k/f_{k+1}| \nearrow 1$, $|g_k/g_{k+1}| \nearrow 1$ *and* $l_{k-1}l_{k+1}/l_k^2 \nearrow +\infty$ *as* $k_0 \leq k \to \infty$ *then*

$$\varliminf_{r\to+\infty} \frac{1}{\beta(\ln r)}\alpha\left(\ln \frac{\mu(r, D^n_{l,[S]}(f*g))}{\mu(r, D^m_{l,[S]}(f*g))}\right) = \varliminf_{k\to\infty} \frac{\alpha(k)}{\beta(\frac{1}{k}\ln\frac{l_k}{l_{k-1}})}. \tag{9}$$

Proof. Using the definitions of the maximal term and the central index, we have

$$\mu(r, D^n_{l,[S]}(f*g)) = \left(\frac{l_1 l_{\nu(r,D^n_{l,[S]}(f*g))-1}}{l_{\nu(r,D^n_{l,[S]}(f*g))}}\right)^n |f_{\nu(r,D^n_{l,[S]}(f*g))}||g_{\nu(r,D^n_{l,[S]}(f*g))}|r^{\nu(r,D^n_{l,[S]}(f*g))} =$$

$$= \left(\frac{l_1 l_{\nu(r,D^n_{l,[S]}(f*g))-1}}{l_{\nu(r,D^n_{l,[S]}(f*g))}}\right)^{n-m} \left(\frac{l_1 l_{\nu(r,D^m_{l,[S]}(f*g))-1}}{l_{\nu(r,D^n_{l,[S]}(f*g))}}\right)^n |f_{\nu(r,D^n_{l,[S]}(f*g))}||g_{\nu(r,D^n_{l,[S]}(f*g))}|r^{\nu(r,D^n_{l,[S]}(f*g))} \leq$$

$$\leq \left(\frac{l_1 l_{\nu(r,D^n_{l,[S]}(f*g))-1}}{l_{\nu(r,D^n_{l,[S]}(f*g))}}\right)^{n-m} \mu(r, D^m_{l,[S]}(f*g))$$

and

$$\mu(r, D^m_{l,[S]}(f*g)) \leq \left(\frac{l_1 l_{\nu(r,D^m_{l,[S]}(f*g))-1}}{l_{\nu(r,D^m_{l,[S]}(f*g))}}\right)^{m-n} \mu(r, D^n_{l,[S]}(f*g)).$$

Therefore, for $m > n$

$$\left(\frac{l_{\nu(r,D^m_{l,[S]}(f*g))}}{l_1 l_{\nu(r,D^m_{l,[S]}(f*g))-1}}\right)^{m-n} \leq \frac{\mu(r, D^n_{l,[S]}(f*g))}{\mu(r, D^m_{l,[S]}(f*g))} \leq \left(\frac{l_{\nu(r,D^n_{l,[S]}(f*g))}}{l_1 l_{\nu(r,D^n_{l,[S]}(f*g))-1}}\right)^{m-n} \tag{10}$$

and

$$\ln\ln \frac{l_{\nu(r,D^m_{l,[S]}(f*g))}}{l_1 l_{\nu(r,D^m_{l,[S]}(f*g))-1}} \leq \ln\ln \frac{\mu(r, D^n_{l,[S]}(f*g))}{\mu(r, D^m_{l,[S]}(f*g))} - \ln(m-n) \leq \ln\ln \frac{l_{\nu(r,D^n_{l,[S]}(f*g))}}{l_1 l_{\nu(r,D^n_{l,[S]}(f*g))-1}}.$$

From condition (7) it follows that $q_1 \ln k \leq \ln\ln \frac{l_k}{l_1 l_{k-1}} \leq Q_1 \ln k$ for every $0 < q_1 < q \leq Q < Q_1 < +\infty$ and all $k \geq k_0(q_1, Q_1)$. Therefore, for all $r \geq r_0$

$$q_1(1+o(1))\ln \nu(r, D^m_{l,[S]}(f*g)) \leq \ln\ln \frac{\mu(r, D^n_{l,[S]}(f*g))}{\mu(r, D^m_{l,[S]}(f*g))} \leq$$
$$\leq Q_1(1+o(1))\ln \nu(r, D^n_{l,[S]}(f*g)), \quad r \to +\infty. \tag{11}$$

Since $\alpha_1 \in L_{si}$, we obtain

$$(1+o(1))\alpha_1(\ln \nu(r, D^m_{l,[S]}(f*g))) \leq \alpha_1\left(\ln\ln \frac{\mu(r, D^n_{l,[S]}(f*g))}{\mu(r, D^m_{l,[S]}(f*g))}\right) \leq$$
$$\leq (1+o(1))\alpha_1(\ln \nu(r, D^n_{l,[S]}(f*g)))$$

as $r\to+\infty$. The condition $\frac{d\beta^{-1}(c\alpha(x))}{d\ln x}=O(1)$ as $x\to+\infty$ for each $c\in(0,+\infty)$ implies that $\alpha(x)=o(\beta(x))$ as $x\to+\infty$. Therefore, by Lemma 3 in view of the condition $\alpha(x)=\alpha_1(\ln x)$ we get

$$\begin{gathered}\varrho_{\alpha,\beta}[D^m_{l,[S]}(f*g))]=\varrho_{\alpha,\beta}[\ln\mu,D^m_{l,[S]}(f*g))]=\varrho_{\alpha,\beta}[\nu,D^m_{l,[S]}(f*g))]\le\\ \le\varlimsup_{r\to+\infty}\frac{1}{\beta(\ln r)}\alpha\left(\ln\frac{\mu(r,D^n_{l,[S]}(f*g))}{\mu(r,D^m_{l,[S]}(f*g))}\right)\le\\ \le\varrho_{\alpha,\beta}[\nu,D^n_{l,[S]}(f*g))]=\varrho_{\alpha,\beta}[\ln\mu,D^n_{l,[S]}(f*g))]=\varrho_{\alpha,\beta}[D^n_{l,[S]}(f*g))]\end{gathered}\tag{12}$$

and

$$\begin{gathered}\lambda_{\alpha,\beta}[D^m_{l,[S]}(f*g))]=\lambda_{\alpha,\beta}[\ln\mu,D^m_{l,[S]}(f*g))]=\lambda_{\alpha,\beta}[\nu,D^m_{l,[S]}(f*g))]\le\\ \le\varlimsup_{r\to+\infty}\frac{1}{\beta(\ln r)}\alpha\left(\ln\frac{\mu(r,D^n_{l,[S]}(f*g))}{\mu(r,D^m_{l,[S]}(f*g))}\right)\le\\ \le\lambda_{\alpha,\beta}[\nu,D^n_{l,[S]}(f*g))]=\lambda_{\alpha,\beta}[\ln\mu,D^n_{l,[S]}(f*g))]=\lambda_{\alpha,\beta}[D^n_{l,[S]}(f*g))].\end{gathered}\tag{13}$$

The assumptions Theorem 1 imply the conditions of Lemma 2. Therefore, if $\varliminf\limits_{k\to\infty}\sqrt[k]{|f_kg_k|}>0$ then from (5) we get

$$\begin{gathered}\varrho_{\alpha,\beta}[D^n_{l,[S]}(f*g))]=\varlimsup_{k\to\infty}\alpha(k)/\beta\left(\frac{1}{k}\left(\ln\left(\frac{l_k}{l_1l_{k-1}}\right)^n+\ln\frac{1}{|f_k||g_k|}\right)\right)=\\ =\varlimsup_{k\to\infty}\alpha(k)/\beta\left(\frac{n}{k}\ln\frac{l_k}{l_{k-1}}+O(1)\right)=\varlimsup_{k\to\infty}\alpha(k)/\beta\left(\frac{1}{k}\ln\frac{l_k}{l_{k-1}}\right),\end{gathered}$$

because $\beta\in L_{si}$. Therefore, (12) implies (8).

If $|f_k/f_{k+1}|\nearrow 1$, $|g_k/g_{k+1}|\nearrow 1$ and $l_{k-1}l_{k+1}/l_k^2\nearrow+\infty$ as $k_0\le k\to\infty$ then $|c_k/c_{k+1}|\nearrow +\infty$ as $k_0\le k\to\infty$, where $c_k=(\frac{l_1l_{k-1}}{l_k})^nf_kg_k$. Therefore, as above by Lemma 2 we obtain

$$\lambda_{\alpha,\beta}[D^n_{l,[S]}(f*g))]=\varliminf_{k\to\infty}\alpha(k)/\beta\left(\frac{1}{k}\ln\frac{l_k}{l_{k-1}}\right),$$

and thus, (13) implies (9). □

Using equality (4), we can obtain various corollaries from Theorem 1. For example, the following statement is true.

Corollary 1. *Let $n\in\mathbb{Z}_+$ and $j\in\mathbb{N}$. If the conditions of Theorem 1 hold then*

$$\varlimsup_{r\to+\infty}\frac{1}{\beta(\ln r)}\alpha\left(\ln\frac{\mu(r,D^n_{l,[S]}f*D^n_{l,[S]}g}{\mu(r,D^{n+j}_{l,[S]}f*D^{n+j}_{l,[S]}g}\right)=\varlimsup_{k\to\infty}\frac{\alpha(k)}{\beta(\frac{1}{k}\ln\frac{l_k}{l_{k-1}})}$$

and by conditions $|f_k/f_{k+1}|\nearrow 1$, $|g_k/g_{k+1}|\nearrow 1$ and $l_{k-1}l_{k+1}/l_k^2\nearrow+\infty$ as $k_0\le k\to\infty$ a similar formula is true for $\varliminf$.

Remark 1. Choosing $\alpha(x)=\ln^+x$ and $\beta(x)=x^+$ from the definitions of $\varrho_{\alpha,\beta}[f]$ and $\lambda_{\alpha,\beta}[f]$ we get the definitions of the order $\varrho[f]=\varlimsup\limits_{r\to+\infty}\frac{\ln\ln M(r,f))}{\ln r}$ and the lower order $\lambda[f]=\varliminf\limits_{r\to+\infty}\frac{\ln\ln M(r,f))}{\ln r}$ for entire function (1). The functions $\alpha(x)=\ln^+x$ and $\beta(x)=x^+$ do not satisfy the hypotheses of Theorem 1.

However, it is known (see for example [15, 16, 17]) that for entire function (1) $\varrho[f] = \overline{\lim\limits_{r\to+\infty}} \frac{\ln\nu(r,f))}{\ln r} = \overline{\lim\limits_{k\to\infty}} \frac{k\ln k}{-\ln|f_k|}$, and if, moreover, $|f_k|/|f_{k+1}| \nearrow +\infty$ as $k_0 \le k \to \infty$ then $\lambda[f] = \underline{\lim\limits_{r\to+\infty}} \frac{\ln\nu(r,f))}{\ln r} = \underline{\lim\limits_{k\to\infty}} \frac{k\ln k}{-\ln|f_k|}$. Therefore, (11) yields

$$q_1\varrho[D^m_{l,[S]}(f*g)] \le \overline{\lim_{r\to+\infty}} \frac{1}{\ln r}\ln\ln\frac{\mu(r,D^n_{l,[S]}(f*g))}{\mu(r,D^m_{l,[S]}(f*g))} \le Q_1\varrho[D^n_{l,[S]}(f*g)].$$

Moreover, as in the proof Theorem 1, we get $\varrho[D^n_{l,[S]}(f*g))] = \overline{\lim\limits_{k\to\infty}} \frac{k\ln k}{\ln(l_k/l_{k-1})}$. Using (11), similar results can be obtained for lower limits. Therefore, by virtue of arbitrariness of q_1 and Q_1 we come to the next statement.

Proposition 1. *Let $m > n \ge 1$, $\underline{\lim\limits_{k\to\infty}} \sqrt[k]{|f_k g_k|} > 0$ and (7) hold. Then*

$$q\,\overline{\lim_{k\to\infty}} \frac{k\ln k}{\ln(l_k/l_{k-1})} \le \overline{\lim_{r\to+\infty}} \frac{1}{\ln r}\ln\ln\frac{\mu(r,D^n_{l,[S]}(f*g))}{\mu(r,D^m_{l,[S]}(f*g))} \le Q\,\overline{\lim_{k\to\infty}} \frac{k\ln k}{\ln(l_k/l_{k-1})},$$

and if, moreover, $|f_k/f_{k+1}| \nearrow 1$, $|g_k/g_{k+1}| \nearrow 1$ and $l_{k-1}l_{k+1}/l_k^2 \nearrow +\infty$ as $k_0 \le k \to \infty$ then

$$q\,\underline{\lim_{k\to\infty}} \frac{k\ln k}{\ln(l_k/l_{k-1})} \le \underline{\lim_{r\to+\infty}} \frac{1}{\ln r}\ln\ln\frac{\mu(r,D^n_{l,[S]}(f*g))}{\mu(r,D^m_{l,[S]}(f*g))} \le Q\,\underline{\lim_{k\to\infty}} \frac{k\ln k}{\ln(l_k/l_{k-1})}.$$

2.2. The case $\lim\limits_{\mathbf{k}\to\infty} \sqrt[k]{l_k/l_{k-1}} = \mathbf{1}$. Unlike entire functions for functions (1) with $R[f] = 1$ the maximal term can be bounded, and in order that $\mu(r,f) \uparrow +\infty$ as $r \uparrow 1$ it is necessary and sufficient that $\overline{\lim\limits_{k\to\infty}} |f_k| = +\infty$. In the sequel, we will consider that $\overline{\lim\limits_{k\to\infty}} |f_k| = +\infty$, $\overline{\lim\limits_{k\to\infty}} |g_k| = +\infty$, $|f_k| > 1$ and $|g_k| > 1$ for all $k \ge k_0$.

For $\alpha \in L$, $\beta \in L$ and the function (1) with $R[f] = 1$ the quantities

$$\varrho^{(1)}_{\alpha,\beta}[f] := \varrho^{(1)}_{\alpha,\beta}[\ln M, f] = \overline{\lim_{r\uparrow 1}} \frac{\alpha(\ln M(r,f))}{\beta(1/(1-r))}, \quad \lambda^{(1)}_{\alpha,\beta}[f] := \lambda^{(1)}_{\alpha,\beta}[\ln M, f] = \underline{\lim_{r\uparrow 1}} \frac{\alpha(\ln M(r,f))}{\beta(1/(1-r))}$$

are called ([18]) the *generalized order and the lower generalized order*, respectively. If here we substitute $\ln\mu(r,f)$ or $\nu(r,f)$ instead of $\ln M(r,f)$ then we obtain the definitions of the quantities $\varrho^{(1)}_{\alpha,\beta}[\ln\mu, f]$, $\lambda^{(1)}_{\alpha,\beta}[\ln\mu, f]$ and $\lambda^{(1)}_{\alpha,\beta}[\nu, f]$, $\lambda^{(1)}_{\alpha,\beta}[\nu, f]$, respectively. The following lemma is true.

Lemma 4. *Let $\alpha \in L_{si}$, $\beta \in L_{si}$ and for each $c \in (0,+\infty)$*

$$\overline{\lim_{x\to+\infty}} \frac{d\ln\beta^{-1}(c\alpha(x))}{d\ln x} < 1, \quad \lim_{x\to+\infty} \frac{\alpha(x/\beta^{-1}(c\alpha(x)))}{\alpha(x)} = 1. \tag{14}$$

Then

$$\varrho^{(1)}_{\alpha,\beta}[f] = \overline{\lim_{k\to\infty}} \frac{\alpha(k)}{\beta(k/\ln|f_k|)}. \tag{15}$$

If, moreover, $|f_k/f_{k+1}| \nearrow 1$ as $k_0 \le k \to \infty$ then

$$\lambda^{(1)}_{\alpha,\beta}[f] = \underline{\lim_{k\to\infty}} \frac{\alpha(k)}{\beta(k/\ln|f_k|)}. \tag{16}$$

Formula (15) is proved in [18], and formula (16) follows from the corresponding formula for Dirichlet series with finite abscissa of absolute convergence proved in [19, 20].

The following lemma also is proved in [14].

Lemma 5. *If $\alpha \in L_{si}$, $\beta \in L_{si}$ and $\alpha(x) = o(\beta(x))$ as $x \to +\infty$ then $\varrho^{(1)}_{\alpha,\beta}[f] = \varrho^{(1)}_{\alpha,\beta}[\ln\mu, f]$ and $\lambda^{(1)}_{\alpha,\beta}[f] = \lambda^{(1)}_{\alpha,\beta}[\ln\mu, f]$. If, moreover, $\alpha(e^x) \in L_{si}$ then $\varrho^{(1)}_{\alpha,\beta}[\ln\mu, f] = \varrho^{(1)}_{\alpha,\beta}[\nu, f]$ and $\lambda^{(1)}_{\alpha,\beta}[\ln\mu, f] = \lambda^{(1)}_{\alpha,\beta}[\nu, f]$.*

The following analog of Theorem 1 is hold.

Theorem 2. *Let $m > n \geq 1$, $\alpha(e^x) \in L_{si}$, $\beta \in L_{si}$ and (12) hold. Suppose that*

$$l_{k-1}/l_k \asymp k, \quad k \to \infty. \tag{17}$$

Then

$$\varlimsup_{r\uparrow 1} \frac{1}{\beta(1/(1-r))}\alpha\left(\sqrt[m-n]{\frac{\mu(r, D^m_{l,[S]}(f*g))}{\mu(r, D^n_{l,[S]}(f*g))}}\right) = \max\{\varrho^{(1)}_{\alpha,\beta}[f], \varrho^{(1)}_{\alpha,\beta}[g]\} \tag{18}$$

and if, moreover, $|f_k/f_{k+1}| \nearrow 1$, $|g_k/g_{k+1}| \nearrow 1$ and $l_{k-1}l_{k+1}/l_k^2 \nearrow 1$ as $k_0 \leq k \to \infty$ then

$$\max\{\lambda^{(1)}_{\alpha,\beta}[f], \lambda^{(1)}_{\alpha,\beta}[g]\} \leq \varliminf_{r\uparrow 1} \frac{1}{\beta(1/(1-r))}\alpha\left(\sqrt[m-n]{\frac{\mu(r, D^m_{l,[S]}(f*g))}{\mu(r, D^n_{l,[S]}(f*g))}}\right) \leq$$
$$\leq \min\{\max\{\lambda^{(1)}_{\alpha,\beta}[f], \varrho^{(1)}_{\alpha,\beta}[g]\}, \max\{\lambda^{(1)}_{\alpha,\beta}[g], \varrho^{(1)}_{\alpha,\beta}[f]\}\}. \tag{19}$$

Proof. From (10) for $m > n$ we get

$$\frac{l_1 l_{\nu(r,D^n_{l,[S]}(f*g))-1}}{l_{\nu(r,D^n_{l,[S]}(f*g))}} \leq \sqrt[m-n]{\frac{\mu(r, D^m_{l,[S]}(f*g))}{\mu(r, D^n_{l,[S]}(f*g))}} \leq \frac{l_1 l_{\nu(r,D^m_{l,[S]}(f*g))-1}}{l_{\nu(r,D^m_{l,[S]}(f*g))}}.$$

Since in view of (17) there exist $0 < q \leq Q < +\infty$ such that $qk \leq l_1 l_{k-1}/l_k \leq Qk$, hence we obtain

$$q\nu(r, D^n_{l,[S]}(f*g)) \leq \sqrt[m-n]{\frac{\mu(r, D^m_{l,[S]}(f*g))}{\mu(r, D^n_{l,[S]}(f*g))}} \leq Q\nu(r, D^m_{l,[S]}(f*g)). \tag{20}$$

We remark that since $\varlimsup\limits_{x\to+\infty} \frac{d\ln y\beta^{-1}(c\alpha(x))}{d\ln x} < 1$ as $x \to +\infty$ for each $c \in (0, +\infty)$, we have $\alpha(x) = o(\beta(x)$ and $\frac{\beta^{-1}(c\alpha(x))\ln x}{x} \to 0$ as $x \to +\infty$ for each $c \in (0, +\infty)$. Therefore, in view of the condition $\alpha(e^x) \in L_{si}$ by Lemma 5 we get

$$\varrho^{(1)}_{\alpha,\beta}[D^n_{l,[S]}(f*g))] = \varrho^{(1)}_{\alpha,\beta}[\ln\mu, D^n_{l,[S]}(f*g))] = \varrho^{(1)}_{\alpha,\beta}[\nu, D^n_{l,[S]}(f*g))] \leq$$
$$\leq \varlimsup_{r\uparrow 1} \frac{1}{\beta(1/(1-r))}\alpha\left(\sqrt[m-n]{\frac{\mu(r, D^m_{l,[S]}(f*g))}{\mu(r, D^n_{l,[S]}(f*g))}}\right) \leq$$
$$\leq \varrho^{(1)}_{\alpha,\beta}[\nu, D^m_{l,[S]}(f*g))] = \varrho^{(1)}_{\alpha,\beta}[\ln\mu, D^m_{l,[S]}(f*g))] = \varrho^{(1)}_{\alpha,\beta}[D^m_{l,[S]}(f*g))] \tag{21}$$

and

$$\lambda^{(1)}_{\alpha,\beta}[D^n_{l,[S]}(f*g))] = \lambda^{(1)}_{\alpha,\beta}[\ln\mu, D^n_{l,[S]}(f*g))] = \lambda^{(1)}_{\alpha,\beta}[\nu, D^n_{l,[S]}(f*g))] \leq$$

$$\le \varliminf_{r\uparrow 1}\frac{1}{\beta(1/(1-r))}\alpha\left(\sqrt[m-n]{\frac{\mu(r,D^m_{l,[S]}(f*g))}{\mu(r,D^n_{l,[S]}(f*g))}}\right)\le$$
$$\le \lambda^{(1)}_{\alpha,\beta}[\nu,D^m_{l,[S]}(f*g))]=\lambda^{(1)}_{\alpha,\beta}[\ln\mu,D^m_{l,[S]}(f*g))]=\lambda^{(1)}_{\alpha,\beta}[D^m_{l,[S]}(f*g))]. \qquad (22)$$

Since $l_1l_{k-1}/l_k\ge qk\ge 1$, $|f_k|>1$ and $|g_k|>1$ for all $k\ge k_0$, we have $(\frac{l_1l_{k-1}}{l_k})^n|f_k||g_k|\ge |f_k|$ and, similarly, $(\frac{l_1l_{k-1}}{l_k})^n|f_k||g_k|\ge|g_k|$ for all $k\ge k_0$. Therefore, $\mu(r,D^n_{l,[S]}(f*g))\ge(1+o(1))\mu(r,f)$ and $\mu(r,D^n_{l,[S]}(f*g))\ge(1+o(1))\mu(r,g)$ as $r\uparrow 1$. Therefore, by Lemma 5 we get

$$\varrho^{(1)}_{\alpha,\beta}[D^n_{l,[S]}(f*g))]\ge\max\{\varrho^{(1)}_{\alpha,\beta}[f],\varrho^{(1)}_{\alpha,\beta}[g]\},\lambda^{(1)}_{\alpha,\beta}[D^n_{l,[S]}(f*g))]\ge\max\{\lambda^{(1)}_{\alpha,\beta}[f],\lambda^{(1)}_{\alpha,\beta}[g]\}. \quad (23)$$

On the other hand, if $\varrho^{(1)}_{\alpha,\beta}[f]<+\infty$ and $\varrho^{(1)}_{\alpha,\beta}[g]<+\infty$ then by Lemma 4 $\ln|f_k|\le\frac{k}{\beta^{-1}(\alpha(k)/\varrho_1)}$ and $\ln|g_k|\le\frac{k}{\beta^{-1}(\alpha(k)/\varrho_2)}$ for every $\varrho_1\in(\varrho^{(1)}_{\alpha,\beta}[f],+\infty)$, $\varrho_2\in(\varrho^{(1)}_{\alpha,\beta}[g],+\infty)$ and all $k\ge k_0$. Therefore, since $\frac{\beta^{-1}(c\alpha(x))\ln x}{x}\to 0$ as $x\to+\infty$ for each $c\in(0,+\infty)$, we obtain

$$n\ln\frac{l_1l_{k-1}}{l_k}+\ln|f_k|+\ln|g_k|\le n\ln(l_1Qk)+\frac{k}{\beta^{-1}(\alpha(k)/\varrho_1)}+\frac{k}{\beta^{-1}(\alpha(k)/\varrho_2)}\le$$
$$\le\frac{(2+o(1))k}{\beta^{-1}(\alpha(k)/\max\{\varrho_1,\varrho_2\})},\quad k\to\infty,$$

whence it follows that $\varrho^{(1)}_{\alpha,\beta}[D^n_{l,[S]}(f*g))]\le\max\{\varrho_1,\varrho_2\}$, i. e. in view of the arbitrariness of ϱ_1 and ϱ_2 we get

$$\varrho^{(1)}_{\alpha,\beta}[D^n_{l,[S]}(f*g))]\le\max\{\varrho^{(1)}_{\alpha,\beta}[f],\varrho^{(1)}_{\alpha,\beta}[g]\}. \qquad (24)$$

Inequalities (21), (23) and (24) yield (18).

If $|f_k/f_{k+1}|\nearrow 1$, $|g_k/g_{k+1}|\nearrow 1$ and $l_{k-1}l_{k+1}/l_k^2\nearrow 1$ as $k_0\le k\to\infty$ then $|c_k/c_{k+1}|\nearrow+\infty$ as $k_0\le k\to\infty$, where $c_k=(\frac{l_1l_{k-1}}{l_k})^nf_kg_k$. Since $\ln|f_{k_j}|\le\frac{k_j}{\beta^{-1}(\alpha(k_j)/\lambda)}$ for every $\lambda\in(\lambda^{(1)}_{\alpha,\beta}[f],+\infty)$ and some sequence $(k_j)\uparrow+\infty$, as above, we have by Lemma 4

$$\lambda^{(1)}_{\alpha,\beta}[D^n_{l,[S]}(f*g))]\le\varliminf_{j\to\infty}\frac{\alpha(k_j)}{\beta(k_j/\ln|f_{k_j}g_{k_j}|)}\le\max\{\lambda,\varrho_2\},$$

in view of the arbitrariness of λ and ϱ_2 we get $\lambda^{(1)}_{\alpha,\beta}[D^n_{l,[S]}(f*g))]\le\max\{\lambda^{(1)}_{\alpha,\beta}[f],\varrho^{(1)}_{\alpha,\beta}[g]\}$. Similarly, $\lambda^{(1)}_{\alpha,\beta}[D^n_{l,[S]}(f*g))]\le\max\{\lambda^{(1)}_{\alpha,\beta}[g],\varrho^{(1)}_{\alpha,\beta}[f]\}$. Therefore,

$$\lambda^{(1)}_{\alpha,\beta}[D^n_{l,[S]}(f*g))]\le\min\{\max\{\lambda^{(1)}_{\alpha,\beta}[f],\varrho^{(1)}_{\alpha,\beta}[g]\},\max\{\lambda^{(1)}_{\alpha,\beta}[g],\varrho^{(1)}_{\alpha,\beta}[f]\}\}. \qquad (25)$$

Inequalities (22), (23) and (25) yield (19). □

Remark 2. Choosing $\alpha(x)=\beta(x)=\ln^+x$ from the definitions of $\varrho^{(1)}_{\alpha,\beta}[f]$ and $\lambda^{(1)}_{\alpha,\beta}[f]$ we get the definitions of the order

$$\varrho^{(1)}[f]=\varlimsup_{r\uparrow 1}\frac{\ln^+\ln M(r,f))}{\ln(1/(1-r))}$$

and the lower order

$$\lambda^{(1)}[f]=\varliminf_{r\uparrow 1}\frac{\ln^+\ln M(r,f))}{\ln(1/(1-r))}$$

for function (1) with $R[f]=1$. The functions $\alpha(x)=\beta(x)=\ln^+x$ do not satisfy the assumptions of Theorem 2.

Now we have ([21])

$$\lambda^{(1)}[f] \le \lambda^{(1)}[\nu, f] \le \lambda^{(1)}[f] + 1, \quad \varrho^{(1)}[f] \le \varrho^{(1)}[\nu, f] \le \varrho^{(1)}[f] + 1.$$

We remark also that ([21, 22])

$$\varrho^{(1)}[f] = \frac{\alpha^*[f]}{1 - \alpha^*[f]}, \quad \alpha^*[f] := \varlimsup_{k\to\infty} \frac{\ln^+ \ln |f_k|}{\ln k},$$

and if $|f_k|/|f_{k+1}| \nearrow 1$ as $k_0 \le k \to \infty$ then

$$\lambda^{(1)}[f] = \frac{\alpha_*[f]}{1 - \alpha_*[f]}, \quad \alpha_*[f] := \varliminf_{k\to\infty} \frac{\ln^+ \ln |f_k|}{\ln k}.$$

From (20) we obtain

$$(m-n)\varrho^{(1)}[D^n_{l,[S]}(f*g)] \le \varlimsup_{r\uparrow 1} \frac{1}{\ln(1/(1-r))} \ln \frac{\mu(r, D^m_{l,[S]}(f*g))}{\mu(r, D^n_{l,[S]}(f*g))} \le$$
$$\le (m-n)(\varrho^{(1)}[D^m_{l,[S]}(f*g)] + 1). \tag{26}$$

and

$$(m-n)\lambda^{(1)}[D^n_{l,[S]}(f*g)] \le \varliminf_{r\uparrow 1} \frac{1}{\ln(1/(1-r))} \ln \frac{\mu(r, D^m_{l,[S]}(f*g))}{\mu(r, D^n_{l,[S]}(f*g))} \le$$
$$\le (m-n)(\lambda^{(1)}[D^m_{l,[S]}(f*g)] + 1). \tag{27}$$

Since $l_1 l_{k-1}/l_k \ge qk \ge 1$, $|f_k| > 1$ and $|g_k| > 1$ for all $k \ge k_0$, we have

$$\alpha^*[D^n_{l,[S]}(f*g)] = \varlimsup_{k\to\infty} \frac{\ln^+(n\ln(l_1 l_{k-1}/l_k) + \ln|f_k| + \ln|g_k|)}{\ln k} \ge$$
$$\ge \varlimsup_{k\to\infty} \frac{\ln^+ \min\{nq\ln k, \ln|f_k|\}}{\ln k} = \alpha^*[f],$$

i. e. $\alpha^*[D^n_{l,[S]}(f*g)] \ge \max\{\alpha^*[f], \alpha^*[f]\}$ and, similarly, $\alpha_*[D^n_{l,S]}(f*g)] \ge \max\{\alpha_*[f], \alpha_*[g]\}$, whence

$$\varrho^{(1)}[D^n_{l,[S]}(f*g)] \ge \frac{\max\{\alpha^*[f], \alpha^*[g]\}}{1 - \max\{\alpha^*[f], \alpha^*[g]\}} = \max\{\varrho^{(1)}[f], \varrho^{(1)}[g])\}$$

and similarly $\lambda^{(1)}[f*g] \ge \max\{\lambda^{(1)}[f], \lambda^{(1)}[g])\}$, provided $|f_k/f_{k+1}| \nearrow 1$, $|g_k/g_{k+1}| \nearrow 1$ and $l_{k-1}l_{k+1}/l_k^2 \nearrow +\infty$ as $k_0 \le k \to \infty$.

On the other hand, $\ln|f_k| \le k^{\alpha_1}$ and $\ln|g_k| \le k^{\alpha_2}$ for every $\alpha_1 \in (\alpha^*[f], 1)$, $\alpha_2 \in (\alpha^*[g], 1)$ and all $k \ge k_0$. Therefore,

$$\alpha^*[D^n_{l,[S]}(f*g)] \le \varlimsup_{k\to\infty} \frac{\ln(nQ\ln k) + \ln^+(k^{\alpha_1} + k^{\alpha_2})}{\ln k} \le \max\{\alpha_1, \alpha_2\},$$

i. e. in view of the arbitrariness of α_1 and α_2 we get $\alpha^*[D^n_{l,[S]}(f*g)] \le \max\{\alpha^*[f], \alpha^*[g]\}$. Similarly we obtain $\alpha_*[D^n_{l,[S]}(f*g)] \le \max\{\alpha_*[f], \alpha_*[g]\}$, whence as above we get

$$\varrho^{(1)}[D^n_{l,[S]}(f*g)] \le \max\{\varrho^{(1)}[f], \varrho^{(1)}[g]\},$$
$$\lambda^{(1)}[D^n_{l,[S]}(f*g)] \le \min\{\max\{\lambda^{(1)}[f], \varrho^{(1)}[g]\}, \max\{\lambda^{(1)}[g], \varrho^{(1)}[f]\}\}.$$

Therefore, in view of (26) and (27) we get the following statement.

Proposition 2. *Let $m > n \geq 1$ and (17) hold. Then*

$$(m-n)\max\{\varrho^{(1)}[f],\varrho^{(1)}[g])\} \leq \overline{\lim_{r\uparrow 1}} \frac{1}{\ln(1/(1-r))} \ln \frac{\mu(r, D^m_{l,[S]}(f*g))}{\mu(r, D^n_{l,[S]}(f*g))} \leq$$
$$\leq (m-n)(\max\{\varrho^{(1)}[f],\varrho^{(1)}[g])\}+1).$$

and if, moreover, $|f_k/f_{k+1}| \nearrow 1$, $|g_k/g_{k+1}| \nearrow 1$ and $l_{k-1}l_{k+1}/l_k^2 \nearrow +\infty$ as $k_0 \leq k \to \infty$ then

$$(m-n)\max\{\lambda^{(1)}[f],\lambda^{(1)}[g])\} \leq \underline{\lim_{r\uparrow 1}} \frac{1}{\ln(1/(1-r))} \ln \frac{\mu(r, D^m_{l,[S]}(f*g))}{\mu(r, D^n_{l,[S]}(f*g))} \leq$$
$$\leq (m-n)(\min\{\max\{\lambda^{(1)}[f],\varrho^{(1)}[g]\},\max\{\lambda^{(1)}[g],\varrho^{(1)}[f]\}\}+1).$$

3. Hadamard composition of two Gelfond-Leont'ev-Ruscheweyh derivatives. Let $f \in H$, $g \in H$ and $n \in \mathbb{N}$. Suppose that $\lim\limits_{k\to\infty} \sqrt[k]{\frac{l_{k-1}}{l_k}} = q$. Then $\sqrt[k]{\frac{l_{k-1}}{l_{n+k-1}}} \to q^n$ as $k \to \infty$ and, as in the proof of Lemma 1, we get the following statement.

Lemma 6. *If $f \in H$, $g \in H$, $n \in \mathbb{N}$ and there exists $\lim\limits_{k\to\infty} \sqrt[k]{l_{k-1}/l_k} = q$ then $R[D^n_{l,[R]}(f*g)] \geq q^{-n}$ and and if, moreover, $\lim\limits_{k\to\infty} \sqrt[k]{|g_k|} = 1$ then $R[D^n_{l,[R]}(f*g)] = q^{-n}$.*

From Lemma 6 it follows that if $\sqrt[k]{l_k/l_{k-1}} \to \infty$ as $k \to \infty$ then $D^n_{l,[R]}(f*g) \in E$, and if $\sqrt[k]{l_k/l_{k-1}} \to 1$ as $k \to \infty$ then $D^n_{l,[R]}(f*g) \in H$. As in Section 1, we consider only these two cases.

3.1. The case $\lim\limits_{k\to\infty} \sqrt[k]{l_k/l_{k-1}} = +\infty$. Since equation (4) is not satisfied for the Hadamard composition of Gelfond-Leont'ev-Ruscheweyhen derivatives, more variants arise in the study of the properties of these compositions. Let us start with an analogue of Theorem 1.

Theorem 3. *Let the functions α and β satisfy the assumptions of Theorem 1. Suppose that $\underline{\lim\limits_{k\to\infty}} \sqrt[k]{|f_k g_k|} > 0$ and (7) holds. Then for $m > n \geq 1$*

$$\overline{\lim_{r\to+\infty}} \frac{1}{\beta(\ln r)} \alpha\left(\ln \frac{\mu(r, D^n_{l,[R]}(f*g))}{\mu(r, D^m_{l,[R]}(f*g))}\right) = \overline{\lim_{k\to\infty}} \frac{\alpha(k)}{\beta(\frac{1}{k}\ln\frac{l_k}{l_{k-1}})}$$

and if, moreover, $|f_k/f_{k+1}| \nearrow 1$, $|g_k/g_{k+1}| \nearrow 1$ and $l_{k-1}l_{k+1}/l_k^2 \nearrow +\infty$ as $k_0 \leq k \to \infty$ then

$$\underline{\lim_{r\to+\infty}} \frac{1}{\beta(\ln r)} \alpha\left(\ln \frac{\mu(r, D^n_{l,[R]}(f*g))}{\mu(r, D^m_{l,[R]}(f*g))}\right) = \underline{\lim_{k\to\infty}} \frac{\alpha(k)}{\beta(\frac{1}{k}\ln\frac{l_k}{l_{k-1}})}.$$

Proof. Using the definitions of the maximal term and the central index, we have

$$\mu(r, D^n_{l,[R]}(f*g)) = \frac{l_n l_{\nu(r,D^n_{l,[R]}(f*g))-1}}{l_{\nu(r,D^n_{l,[R]}(f*g))+n-1}} |f_{\nu(r,D^n_{l,[R]}(f*g)}||g_{\nu(r,D^n_{l,[R]}(f*g))}| r^{\nu(r,D^n_{l,[R]}(f*g))} =$$
$$= \frac{l_n l_{\nu(r,D^n_{l,[R]}(f*g))+m-1}}{l_m l_{\nu(r,D^n_{l,[R]}(f*g))+n-1}} \frac{l_m l_{\nu(r,D^n_{l,[R]}(f*g))-1}}{l_{\nu(r,D^n_{l,[R]}(f*g))+m-1}} |f_{\nu(r,D^n_{l,[S]}(f*g)}||g_{\nu(r,D^n_{l,[S]}(f*g))}| r^{\nu(r,D^n_{l,[S]}(f*g))} \leq$$

$$\leq \frac{l_n l_{\nu(r,D^n_{l,[R]}(f*g))+m-1}}{l_m l_{\nu(r,D^n_{l,[R]}(f*g))+n-1}} \mu(r, D^m_{l,[R]}(f*g)),$$

$$\mu(r, D^m_{l,[R]}(f*g)) \leq \frac{l_m l_{\nu(r,D^m_{l,[R]}(f*g))+n-1}}{l_n l_{\nu(r,D^m_{l,[R]}(f*g))+m-1}} \mu(r, D^n_{l,[R]}(f*g)).$$

Thus,

$$\frac{l_n l_{\nu(r,D^m_{l,[R]}(f*g))+m-1}}{l_m l_{\nu(r,D^m_{l,[R]}(f*g))+n-1}} \leq \frac{\mu(r, D^n_{l,[R]}(f*g))}{\mu(r, D^m_{l,[R]}(f*g))} \leq \frac{l_n l_{\nu(r,D^n_{l,[R]}(f*g))+m-1}}{l_m l_{\nu(r,D^n_{l,[R]}(f*g))+n-1}}. \tag{28}$$

In view of (7) for every $0 < q_1 < q \leq Q < Q_1 < +\infty$ we have $k^{q_1} \leq \ln l_k - \ln l_{k-1} \leq k^{Q_1}$ for all $k \geq k_0(q_1, Q_1)$. Therefore,

$$\begin{gathered} \ln l_{\nu(r,D^n_{l,[R]}(f*g))+m-1} - \ln l_{\nu(r,D^n_{l,[R]}(f*g))+n-1} = \\ = \ln l_{\nu(r,D^n_{l,[R]}(f*g))+m-1} - \ln l_{\nu(r,D^n_{l,[R]}(f*g))+m-2} + \dots + \\ + \ln l_{\nu(r,D^n_{l,[R]}(f*g))+m-(m-n)} - \ln l_{\nu(r,D^n_{l,[R]}(f*g))+m-(m-n)-1} \leq \\ \leq (\nu(r, D^n_{l,[R]}(f*g)) + m - 1)^{Q_1} + \dots + (\nu(r, D^n_{l,[R]}(f*g)) + m - (m-n))^{Q_1} = \\ = (m-n)(1+o(1))\nu(r, D^n_{l,[R]}(f*g))^{Q_1}, \quad r \to +\infty. \end{gathered}$$

Similarly, as $r \to +\infty$

$$\ln l_{\nu(r,D^n_{l,[R]}(f*g))+m-1} - \ln l_{\nu(r,D^n_{l,[R]}(f*g))+n-1} \geq (m-n)(1+o(1))\nu(r, D^n_{l,[R]}(f*g))^{q_1}.$$

Therefore, from (28) we obtain (11) with $D_{l,[R]}$ instead of $D_{l,[S]}$. Inequalities (11) imply (12) and (13) with $D_{l,[R]}$ instead of $D_{l,[S]}$. Finally, if $\varliminf\limits_{k\to\infty} \sqrt[k]{|f_k g_k|} > 0$ then by Lemma 2

$$\varrho_{\alpha,\beta}[D^n_{l,[R]}(f*g))] = \varlimsup_{k\to\infty} \alpha(k)/\beta\left(\frac{1}{k}\ln\frac{l_{n+k-1}}{l_1 l_{k-1}} + \ln\frac{1}{|f_k||g_k|}\right) = \varlimsup_{k\to\infty} \alpha(k)/\beta\left(\frac{1}{k}\ln\frac{l_k}{l_{k-1}}\right),$$

and, similarly,

$$\lambda_{\alpha,\beta}[D^n_{l,[R]}(f*g))] = \varliminf_{k\to\infty} \alpha(k)/\beta\left(\frac{1}{k}\ln\frac{l_k}{l_{k-1}}\right).$$

□

Remark 3. Using the proof of Proposition 1 we get the following statement.

Proposition 3. *Let $m > n \geq 1$, $\varliminf\limits_{k\to\infty} \sqrt[k]{|f_k g_k|} > 0$ and (7) hold. Then*

$$q \varlimsup_{k\to\infty} \frac{k\ln k}{\ln(l_k/l_{k-1})} \leq \varlimsup_{r\to+\infty} \frac{1}{\ln r} \ln\ln \frac{\mu(r, D^n_{l,[R]}(f*g))}{\mu(r, D^m_{l,[R]}(f*g))} \leq Q \varlimsup_{k\to\infty} \frac{k\ln k}{\ln(l_k/l_{k-1})},$$

and if, moreover, $|f_k/f_{k+1}| \nearrow 1$, $|g_k/g_{k+1}| \nearrow 1$ and $l_{k-1}l_{k+1}/l_k^2 \nearrow +\infty$ as $k_0 \leq k \to \infty$ then

$$q \varliminf_{k\to\infty} \frac{k\ln k}{\ln(l_k/l_{k-1})} \leq \varliminf_{r\to+\infty} \frac{1}{\ln r} \ln\ln \frac{\mu(r, D^n_{l,[R]}(f*g))}{\mu(r, D^m_{l,[R]}(f*g))} \leq Q \varliminf_{k\to\infty} \frac{k\ln k}{\ln(l_k/l_{k-1})},$$

The following theorem is an analogue of Corollary 1.

Theorem 4. *Let the functions α and β satisfy the conditions of Theorem 1, $n\in\mathbb{N}$ and $j\in\mathbb{N}$. Suppose that $\varliminf\limits_{k\to\infty}\sqrt[k]{|f_kg_k|}>0$ and (7) holds. Then*

$$\varlimsup_{r\to+\infty}\frac{1}{\beta(\ln r)}\alpha\left(\ln\frac{\mu(r,D^n_{l,[R]}f*D^n_{l,[R]}g)}{\mu(r,D^{n+j}_{l,[R]}f*D^{n+j}_{l,[R]}g)}\right)=\varlimsup_{k\to\infty}\frac{\alpha(k)}{\beta(\frac{1}{k}\ln\frac{l_k}{l_{k-1}})}$$

and if, moreover, $|f_k/f_{k+1}|\nearrow 1$, $|g_k/g_{k+1}|\nearrow 1$ and $l_{k-1}l_{k+1}/l_k^2\nearrow+\infty$ as $k_0\le k\to\infty$ then

$$\varliminf_{r\to+\infty}\frac{1}{\beta(\ln r)}\alpha\left(\ln\frac{\mu(r,D^n_{l,[R]}f*D^n_{l,[R]}g)}{\mu(r,D^{n+j}_{l,[R]}f*D^{n+j}_{l,[R]}g)}\right)=\varliminf_{k\to\infty}\frac{\alpha(k)}{\beta(\frac{1}{k}\ln\frac{l_k}{l_{k-1}})}.$$

Proof. Since $(D^n_{l,[R]}f*D^n_{l,[R]}g)(z)=z+\sum\limits_{k=2}^{\infty}\left(\frac{l_{k-1}l_n}{l_{n+k-1}}\right)^2 f_kg_kz^k$, we have

$$\mu(r,D^n_{l,[R]}f*D^n_{l,[R]}g)=$$
$$=\left(\frac{l_{\nu(r,D^n_{l,[R]}f*D^n_{l,[R]}g)-1}l_n}{l_{n+\nu(r,D^n_{l,[R]}f*D^n_{l,[R]}g)-1}}\right)^2|f_{\nu(r,D^n_{l,[R]}f*D^n_{l,[R]}g)}||g_{\nu(r,D^n_{l,[R]}f*D^n_{l,[R]}g)}|r^{\nu(r,D^n_{l,[R]}f*D^n_{l,[R]}g)}=$$
$$=\left(\frac{l_nl_{\nu(r,D^n_{l,[R]}f*D^n_{l,[R]}g)+n+j-1}}{l_{n+j}l_{\nu(r,D^n_{l,[R]}f*D^n_{l,[R]}g)+n-1}}\right)^2\left(\frac{l_{n+j}l_{\nu(r,D^n_{l,[R]}f*D^n_{l,[R]}g)-1}}{l_{\nu(r,D^n_{l,[R]}f*D^n_{l,[R]}g)+n+j-1}}\right)^2\times$$
$$\times|f_{\nu(r,D^n_{l,[R]}f*D^n_{l,[R]}g)}||g_{\nu(r,D^n_{l,[R]}f*D^n_{l,[R]}g)}|r^{\nu(r,D^n_{l,[R]}f*D^n_{l,[R]}g)}\le$$
$$\le\left(\frac{l_nl_{\nu(r,D^n_{l,[R]}f*D^n_{l,[R]}g)+n+j-1}}{l_{n+j}l_{\nu(r,D^n_{l,[R]}f*D^n_{l,[R]}g)+n-1}}\right)^2\mu(r,D^{n+j}_{l,[R]}f*D^{n+j}_{l,[R]}g)$$

and, similarly,

$$\mu(r,D^{n+j}_{l,[R]}f*D^{n+j}_{l,[R]}g)\le\left(\frac{l_{n+j}l_{\nu(r,D^{n+j}_{l,[R]}f*D^{n+j}_{l,[R]}g)+n-1}}{l_nl_{\nu(r,D^{n+j}_{l,[R]}f*D^{n+j}_{l,[R]}g)+n+j-1}}\right)^2\mu(r,D^n_{l,[R]}f*D^n_{l,[R]}g).$$

Thus,

$$\left(\frac{l_nl_{\nu(r,D^{n+j}_{l,[R]}f*D^{n+j}_{l,[R]}g)+n+j-1}}{l_{n+j}l_{\nu(r,D^{n+j}_{l,[R]}f*D^{n+j}_{l,[R]}g)+n-1}}\right)^2\le\frac{\mu(r,D^n_{l,[R]}f*D^n_{l,[R]}g)}{\mu(r,D^{n+j}_{l,[R]}f*D^{n+j}_{l,[R]}g)}\le\left(\frac{l_nl_{\nu(r,D^n_{l,[R]}f*D^n_{l,[R]}g)+n+j-1}}{l_{n+j}l_{\nu(r,D^n_{l,[R]}f*D^n_{l,[R]}g)+n-1}}\right)^2. \quad (29)$$

Since in view of (7) $k^{q_1}\le\ln l_k-\ln l_{k-1}\le k^{Q_1}$ for every $0<q_1<q\le Q<Q_1<+\infty$ and all $k\ge k_0(q_1,Q_1)$, as in the proof of Theorem 3, we obtain

$$(1+o(1))j\nu(r,D^n_{l,[R]}f*D^n_{l,[R]}g)^{q_1}\le\ln l_{\nu(r,D^n_{l,[R]}f*D^n_{l,[R]}g)+n+j-1}-\ln l_{\nu(r,D^n_{l,[R]}f*D^n_{l,[R]}g)+n-1}\le$$
$$\le(1+o(1))j\nu(r,D^n_{l,[R]}f*D^n_{l,[R]}g)^{Q_1},\quad r\to+\infty.$$

Therefore, (29) yields

$$(1+o(1))q_1\ln\nu(r,D^{n+j}_{l,[R]}f*D^{n+j}_{l,[R]}g))\le\ln\ln\frac{\mu(r,D^n_{l,[R]}f*D^n_{l,[R]}g)}{\mu(r,D^{n+j}_{l,[R]}f*D^{n+j}_{l,[R]}g)}\le$$

$$\leq (1+o(1))Q_1 \ln \nu(r, D^n_{l,[R]} f * D^n_{l,[R]} g), \quad r \to +\infty, \tag{30}$$

i. e. we obtain (11) with $D^n_{l,[R]} f * D^n_{l,[R]} g$ instead of $D^n_{l,[S]}(f*g)$. This inequalities imply (12) and (13) with $D^n_{l,[R]} f * D^n_{l,[R]} g$ instead of $D^n_{l,[S]}(f*g)$. Finally, if $\varliminf\limits_{k\to\infty} \sqrt[k]{|f_k g_k|} > 0$ then by Lemma 2

$$\varrho_{\alpha,\beta}[D^n_{l,[R]} f * D^n_{l,[R]} g] = \varlimsup_{k\to\infty} \alpha(k)/\beta\left(\frac{2}{k}\ln\frac{l_{n+k-1}}{l_1 l_{k-1}} + \ln\frac{1}{|f_k||g_k|}\right) = \varlimsup_{k\to\infty} \alpha(k)/\beta\left(\frac{1}{k}\ln\frac{l_k}{l_{k-1}}\right),$$

and, similarly,

$$\lambda_{\alpha,\beta}[D^n_{l,[R]} f * D^n_{l,[R]} g] = \varliminf_{k\to\infty} \alpha(k)/\beta\left(\frac{1}{k}\ln\frac{l_k}{l_{k-1}}\right).$$

□

Remark 4. Using (30), as in the proof of Proposition 1 we get the following statement.

Proposition 4. *Let $n \in \mathbb{N}$, $j \in \mathbb{N}$, $\varliminf\limits_{k\to\infty} \sqrt[k]{|f_k g_k|} > 0$ and (7) hold. Then*

$$q \varlimsup_{k\to\infty} \frac{k\ln k}{\ln(l_k/l_{k-1})} \leq \varlimsup_{r\to+\infty} \frac{1}{\ln r} \ln\ln \frac{\mu(r, D^n_{l,[R]} f * D^n_{l,[R]} g)}{\mu(r, D^{n+j}_{l,[R]} f * D^{n+j}_{l,[R]} g)} \leq Q \varlimsup_{k\to\infty} \frac{k\ln k}{\ln(l_k/l_{k-1})},$$

and if, moreover, $|f_k/f_{k+1}| \nearrow 1$, $|g_k/g_{k+1}| \nearrow 1$ and $l_{k-1}l_{k+1}/l_k^2 \nearrow +\infty$ as $k_0 \leq k \to \infty$ then

$$q \varliminf_{k\to\infty} \frac{k\ln k}{\ln(l_k/l_{k-1})} \leq \varliminf_{r\to+\infty} \frac{1}{\ln r} \ln\ln \frac{\mu(r, D^n_{l,[R]} f * D^n_{l,[R]} g)}{\mu(r, D^{n+j}_{l,[R]} f * D^{n+j}_{l,[R]} g)} \leq Q \varliminf_{k\to\infty} \frac{k\ln k}{\ln(l_k/l_{k-1})}.$$

Theorem 5. *Let the functions α and β satisfy the conditions of Theorem 1, $n \in \mathbb{N}$ and $j \in \mathbb{N}$. Suppose that $\varliminf\limits_{k\to\infty} \sqrt[k]{|f_k g_k|} > 0$ and (7) holds. Then*

$$\varlimsup_{r\to+\infty} \frac{1}{\beta(\ln r)} \alpha\left(\ln \frac{\mu(r, D^n_{l,[R]}(f*g))}{\mu(r, D^{n+j}_{l,[R]} f * D^{n+j}_{l,[R]} g)}\right) = \varlimsup_{k\to\infty} \frac{\alpha(k)}{\beta(\frac{1}{k}\ln\frac{l_k}{l_{k-1}})}$$

and if, moreover, $|f_k/f_{k+1}| \nearrow 1$, $|g_k/g_{k+1}| \nearrow 1$ and $l_{k-1}l_{k+1}/l_k^2 \nearrow +\infty$ as $k_0 \leq k \to \infty$ then

$$\varliminf_{r\to+\infty} \frac{1}{\beta(\ln r)} \alpha\left(\ln \frac{\mu(r, D^n_{l,[R]}(f*g))}{\mu(r, D^{n+j}_{l,[R]} f * D^{n+j}_{l,[R]} g)}\right) = \varliminf_{k\to\infty} \frac{\alpha(k)}{\beta(\frac{1}{k}\ln\frac{l_k}{l_{k-1}})}.$$

Proof. As above, we have

$$\mu(r, D^n_{l,[R]}(f*g)) = \frac{l_n l_{\nu(r,D^n_{l,[R]}(f*g))-1}}{l_{\nu(r,D^n_{l,[R]}(f*g))+n-1}} |f_{\nu(r,D^n_{l,[R]}(f*g)}||g_{\nu(r,D^n_{l,[R]}(f*g))}| r^{\nu(r,D^n_{l,[R]}(f*g))} =$$

$$= \frac{l_n l^2_{\nu(r,D^n_{l,[R]}(f*g))+n+j-1}}{l^2_{n+j} l_{\nu(r,D^n_{l,[R]}(f*g))+n-1} l_{\nu(r,D^n_{l,[R]}(f*g))-1}} \left(\frac{l_{n+j} l_{\nu(r,D^n_{l,[R]}(f*g))-1}}{l_{\nu(r,D^n_{l,[R]}(f*g))+n+j-1}}\right)^2 \times$$

$$\times |f_{\nu(r,D^n_{l,[R]}(f*g)}||g_{\nu(r,D^n_{l,[R]}(f*g))}| r^{\nu(r,D^n_{l,[R]}(f*g))} \leq$$

$$\leq \frac{l_n l^2_{\nu(r,D^n_{l,[R]}(f*g))+n+j-1}}{l^2_{n+j} l_{\nu(r,D^n_{l,[R]}(f*g))+n-1} l_{\nu(r,D^n_{l,[R]}(f*g))-1}} \mu(r, D^{n+j}_{l,[R]} f * D^{n+j}_{l,[R]} g).$$

On the other hand,

$$\mu(r, D^{n+j}_{l,[R]} f * D^{n+j}_{l,[R]} g) = \left(\frac{l_{n+j} l_{\nu(r,D^{n+j}_{l,[R]} f * D^{n+j}_{l,[R]} g)-1}}{l_{\nu(r,\nu(r,D^{n+j}_{l,[R]} f * D^{n+j}_{l,[R]} g)+n+j-1}} \right)^2 \times$$

$$\times |f_{\nu(r,D^{n+j}_{l,[R]} f * D^{n+j}_{l,[R]} g)}||g_{\nu(r,D^{n+j}_{l,[R]} f * D^{n+j}_{l,[R]} g)}| r^{\nu(r,D^{n+j}_{l,[R]} f * D^{n+j}_{l,[R]} g)} =$$

$$= \frac{l^2_{n+j} l_{\nu(r,D^{n+j}_{l,[R]} f * D^{n+j}_{l,[R]} g)-1} l_{\nu(r,D^{n+j}_{l,[R]} f * D^{n+j}_{l,[R]} g)+n-1}}{l^2_{\nu(r,D^{n+j}_{l,[R]} f * D^{n+j}_{l,[R]} g)+n+j-1}} \frac{l_n l_{\nu(r,D^{n+j}_{l,[R]} f * D^{n+j}_{l,[R]} g)-1}}{l_{\nu(r,D^{n+j}_{l,[R]} f * D^{n+j}_{l,[R]} g)+n-1}} \times$$

$$\times |f_{\nu(r,D^{n+j}_{l,[R]} f * D^{n+j}_{l,[R]} g)}||g_{\nu(r,D^{n+j}_{l,[R]} f * D^{n+j}_{l,[R]} g)}| r^{\nu(r,D^{n+j}_{l,[R]} f * D^{n+j}_{l,[R]} g)} \leq$$

$$\leq \frac{l^2_{n+j} l_{\nu(r,D^{n+j}_{l,[R]} f * D^{n+j}_{l,[R]} g)-1} l_{\nu(r,D^{n+j}_{l,[R]} f * D^{n+j}_{l,[R]} g)+n-1}}{l_n l^2_{\nu(r,D^{n+j}_{l,[R]} f * D^{n+j}_{l,[R]} g)+n+j-1}} \mu(r, D^n_{l,[R]}(f * g)).$$

Therefore,

$$\frac{l_n l^2_{\nu(r,D^{n+j}_{l,[R]} f * D^{n+j}_{l,[R]} g)+n+j-1}}{l^2_{n+j} l_{\nu(r,D^{n+j}_{l,[R]} f * D^{n+j}_{l,[R]} g)-1} l_{\nu(r,D^{n+j}_{l,[R]} f * D^{n+j}_{l,[R]} g)+n-1}} \leq$$

$$\leq \frac{\mu(r, D^n_{l,[R]}(f * g))}{\mu(r, D^{n+j}_{l,[R]} f * D^{n+j}_{l,[R]} g)} \leq \frac{l_n l^2_{\nu(r,D^n_{l,[R]}(f*g))+n+j-1}}{l^2_{n+j} l_{\nu(r,D^n_{l,[R]}(f*g))+n-1} l_{\nu(r,D^n_{l,[R]}(f*g))-1}} \tag{31}$$

As in the proof of Theorem 3 we have

$$\ln \frac{l^2_{\nu(r,D^n_{l,[R]}(f*g))+n+j-1}}{l_{\nu(r,D^n_{l,[R]}(f*g))+n-1} l_{\nu(r,D^n_{l,[R]}(f*g))-1}} = \ln l_{\nu(r,D^n_{l,[R]}(f*g))+n+j-1} - \ln l_{\nu(r,D^n_{l,[R]}(f*g))+n-1} +$$

$$+ \ln l_{\nu(r,D^n_{l,[R]}(f*g))+n+j-1} - \ln l_{\nu(r,D^n_{l,[R]}(f*g))-1} \leq (n+2j)(1+o(1))\nu(r, D^n_{l,[R]}(f * g)^{Q_1}$$

and, similarly,

$$\ln \frac{l^2_{\nu(r,D^{n+j}_{l,[R]} f * D^{n+j}_{l,[R]} g)+n+j-1}}{l_{\nu(r,D^{n+j}_{l,[R]} f * D^{n+j}_{l,[R]} g)-1} l_{\nu(r,D^{n+j}_{l,[R]} f * D^{n+j}_{l,[R]} g)+n-1}} \geq (n+2j)(1+o(1))\nu(r, D^{n+j}_{l,[R]} f * D^{n+j}_{l,[R]} g)^{q_1}$$

as $r \to +\infty$. Therefore, (31) implies

$$(1+o(1))q_1 \ln \nu(r, D^{n+j}_{l,[R]} f * D^{n+j}_{l,[R]} g) \leq \ln \ln \frac{\mu(r, D^n_{l,[R]}(f * g))}{\mu(r, D^{n+j}_{l,[R]} f * D^{n+j}_{l,[R]} g)} \leq$$

$$\leq (1+o(1))Q_1 \ln \nu(r, D^n_{l,[R]}(f * g), \quad r \to +\infty. \tag{32}$$

□

The further proof of Theorem 5 is the same as the proof of Theorems 3 and 4.

Remark 5. Using (32), as above we get the following statement.

Proposition 5. *Let* $n\in\mathbb{N}$, $j\in\mathbb{N}$, $\varliminf\limits_{k\to\infty}\sqrt[k]{|f_kg_k|}>0$ *and* (7) *hold. Then*

$$q\varlimsup_{k\to\infty}\frac{k\ln k}{\ln(l_k/l_{k-1})}\le\varlimsup_{r\to+\infty}\frac{1}{\ln r}\ln\ln\frac{\mu(r,D^n_{l,[R]}(f*g))}{\mu(r,D^{n+j}_{l,[R]}f*D^{n+j}_{l,[R]}g)}\le Q\varlimsup_{k\to\infty}\frac{k\ln k}{\ln(l_k/l_{k-1})},$$

and if, moreover, $|f_k/f_{k+1}|\nearrow 1$, $|g_k/g_{k+1}|\nearrow 1$ *and* $l_{k-1}l_{k+1}/l_k^2\nearrow+\infty$ *as* $k_0\le k\to\infty$ *then*

$$q\varliminf_{k\to\infty}\frac{k\ln k}{\ln(l_k/l_{k-1})}\le\varliminf_{r\to+\infty}\frac{1}{\ln r}\ln\ln\frac{\mu(r,D^n_{l,[R]}(f*g))}{\mu(r,D^{n+j}_{l,[R]}f*D^{n+j}_{l,[R]}g)}\le Q\varliminf_{k\to\infty}\frac{k\ln k}{\ln(l_k/l_{k-1})}.$$

3.2. The case $\lim\limits_{k\to\infty}\sqrt[k]{l_k/l_{k-1}}=1$. As above, we will consider that

$$\varlimsup_{k\to\infty}|f_k|=+\infty,\ \varlimsup_{k\to\infty}|g_k|=+\infty,\ |f_k|>1 \text{ and } |g_k|>1 \text{ for all } k\ge k_0.$$

Theorem 6. *Let* $m>n\ge1$ *and the functions* α *and* β *satisfy the conditions of Theorem 2. Suppose that* (17) *holds. Then*

$$\varlimsup_{r\uparrow1}\frac{1}{\beta(1/(1-r))}\alpha\left(\sqrt[m-n]{\frac{\mu(r,D^m_{l,[R]}(f*g))}{\mu(r,D^n_{l,[R]}(f*g))}}\right)=\max\{\varrho^{(1)}_{\alpha,\beta}[f],\varrho^{(1)}_{\alpha,\beta}[g]\}$$

and if, moreover, $|f_k/f_{k+1}|\nearrow 1$, $|g_k/g_{k+1}|\nearrow 1$ *and* $l_{k-1}l_{k+1}/l_k^2\nearrow 1$ *as* $k_0\le k\to\infty$ *then*

$$\max\{\lambda^{(1)}_{\alpha,\beta}[f],\lambda^{(1)}_{\alpha,\beta}[g]\}\le\varlimsup_{r\uparrow1}\frac{1}{\beta(1/(1-r))}\alpha\left(\sqrt[m-n]{\frac{\mu(r,D^m_{l,[R]}(f*g))}{\mu(r,D^n_{l,[R]}(f*g))}}\right)\le$$
$$\le\min\{\max\{\lambda^{(1)}_{\alpha,\beta}[f],\varrho^{(1)}_{\alpha,\beta}[g]\},\max\{\lambda^{(1)}_{\alpha,\beta}[g],\varrho^{(1)}_{\alpha,\beta}[f]\}\}.$$

Proof. In view of (17) there exist $0<q\le Q<+\infty$ such that $qk\le l_{k-1}/l_k\le Qk$. From (28) we obtain

$$\frac{l_ml_{\nu(r,D^n_{l,[R]}(f*g))+n-1}}{l_nl_{\nu(r,D^n_{l,[R]}(f*g))+m-1}}\le\frac{\mu(r,D^m_{l,[R]}(f*g))}{\mu(r,D^n_{l,[R]}(f*g))}\le\frac{l_ml_{\nu(r,D^m_{l,[R]}(f*g))+n-1}}{l_nl_{\nu(r,D^m_{l,[R]}(f*g))+m-1}},$$

whence

$$(1+o(1))\frac{l_m}{l_n}(q\nu(r,D^n_{l,[R]}(f*g)))^{(m-n)}\le\frac{\mu(r,D^m_{l,[R]}(f*g))}{\mu(r,D^n_{l,[R]}(f*g))}\le$$
$$\le(1+o(1))\frac{l_m}{l_n}(Q\nu(r,D^m_{l,[R]}(f*g)))^{(m-n)},\quad r\uparrow1,$$
$$q_1\nu(r,D^n_{l,[R]}(f*g))\le\sqrt[m-n]{\frac{\mu(r,D^m_{l,[R]}(f*g))}{\mu(r,D^n_{l,[R]}(f*g))}}\le Q_1\nu(r,D^m_{l,[R]}(f*g))\tag{33}$$

for some $0<q_1\le Q_1<+\infty$ and $r\in[r_0,1)$, that is (20) holds with q_1,Q_1 instead of q,Q and $D^n_{l,[R]}(f*g)$ instead of $D^n_{l,[S]}(f*g)$. Therefore, (21) and (22) hold with $D^n_{l,[R]}(f*g)$ instead of $D^n_{l,[S]}(f*g)$.

Since $l_{k-1}/l_k \geq qk \geq 1$, $|f_k| > 1$ and $|g_k| > 1$ for all $k \geq k_0$, we have $\frac{l_n l_{k-1}}{l_{k+n-1}}|f_k||g_k| \geq l_n(qk)^n|f_k| \geq |f_k|$ and, similarly, $\frac{l_n l_{k-1}}{l_{k+n-1}}|f_k||g_k| \geq |g_k|$ for all $k \geq k_0$. Therefore, as in the proof of Theorem 2 we get (23) with $D^n_{l,[R]}(f*g)$ instead of $D^n_{l,[S]}(f*g)$.

On the other hand, for $\varrho_1 > \varrho^{(1)}_{\alpha,\beta}[f]$ and $\varrho_2 > \varrho^{(1)}_{\alpha,\beta}[g]$, as in the proof of Theorem 2 we have

$$\ln\frac{l_n l_{k-1}}{l_{k+n-1}} + \ln|f_k| + \ln|g_k| \leq \ln(l_n(Qk)^n) + \frac{k}{\beta^{-1}(\alpha(k)/\varrho_1)} + \frac{k}{\beta^{-1}(\alpha(k)/\varrho_2)} \leq$$
$$\leq \frac{(2+o(1))k}{\beta^{-1}(\alpha(k)/\max\{\varrho_1,\varrho_2\})}, \quad k\to\infty,$$

whence, as above we get (24) and, similarly, (25) with $D^n_{l,[R]}(f*g)$ instead of $D^n_{l,[S]}(f*g)$. Thus, in (18) and (19) you can put $D^n_{l,[R]}(f*g)$ instead of $D^n_{l,[S]}(f*g)$. □

Remark 6. For the usual orders the following proposition is true.

Proposition 6. *Let $m > n \geq 1$ and (17) hold. Then*

$$(m-n)\max\{\varrho^{(1)}[f],\varrho^{(1)}[g])\} \leq \varlimsup_{r\uparrow 1}\frac{1}{\ln(1/(1-r))}\ln\frac{\mu(r,D^m_{l,[R]}(f*g))}{\mu(r,D^n_{l,[R]}(f*g))} \leq$$
$$\leq (m-n)(\max\{\varrho^{(1)}[f],\varrho^{(1)}[g])\}+1).$$

and if, moreover, $|f_k/f_{k+1}| \nearrow 1$, $|g_k/g_{k+1}| \nearrow 1$ and $l_{k-1}l_{k+1}/l_k^2 \nearrow +\infty$ as $k_0 \leq k \to \infty$ then

$$(m-n)\max\{\lambda^{(1)}[f],\lambda^{(1)}[g])\} \leq \varliminf_{r\uparrow 1}\frac{1}{\ln(1/(1-r))}\ln\frac{\mu(r,D^m_{l,[R]}(f*g))}{\mu(r,D^n_{l,[R]}(f*g))} \leq$$
$$\leq (m-n)(\min\{\max\{\lambda^{(1)}[f],\varrho^{(1)}[g]\},\max\{\lambda^{(1)}[g],\varrho^{(1)}[f]\}\}+1).$$

The proof of Proposition 6 is the same as that proof of Proposition 2. We just note that, since $\ln(l_n qk)^n \leq \frac{l_n l_{k-1}}{l_{k+n-1}} \leq \ln(l_n Qk)^n$, we get $\alpha^*[D^n_{l,[R]}(f*g)] = \max\{\alpha^*[f],\alpha^*[f]\}$ and $\alpha_*[D^n_{l,[R]}(f*g)] = \max\{\alpha_*[f],\alpha_*[g]\}$.

The following theorem holds for $D^n_{l,[R]}f * D^n_{l,[R]}g)$.

Theorem 7. *Let the functions α and β satisfy the assumptions of Theorem 2, $n\in\mathbb{N}$ and $j\in\mathbb{N}$. Suppose that (17) holds. Then*

$$\varlimsup_{r\uparrow 1}\frac{1}{\beta(1/(1-r))}\alpha\left(\sqrt[2j]{\frac{\mu(r,D^{n+j}_{l,[R]}f*D^{n+j}_{l,[R]}g)}{\mu(r,D^n_{l,[R]}f*D^n_{l,[R]}g)}}\right) = \max\{\varrho^{(1)}_{\alpha,\beta}[f],\varrho^{(1)}_{\alpha,\beta}[g]\}$$

and if, moreover, $|f_k/f_{k+1}| \nearrow 1$, $|g_k/g_{k+1}| \nearrow 1$ and $l_{k-1}l_{k+1}/l_k^2 \nearrow 1$ as $k_0 \leq k \to \infty$ then

$$\max\{\lambda^{(1)}_{\alpha,\beta}[f],\lambda^{(1)}_{\alpha,\beta}[g]\} \leq \varlimsup_{r\uparrow 1}\frac{1}{\beta(1/(1-r))}\alpha\left(\sqrt[2j]{\frac{\mu(r,D^{n+j}_{l,[R]}f*D^{n+j}_{l,[R]}g)}{\mu(r,D^n_{l,[R]}f*D^n_{l,[R]}g)}}\right) \leq$$
$$\leq \min\{\max\{\lambda^{(1)}_{\alpha,\beta}[f],\varrho^{(1)}_{\alpha,\beta}[g]\},\max\{\lambda^{(1)}_{\alpha,\beta}[g],\varrho^{(1)}_{\alpha,\beta}[f]\}\}.$$

Proof. From (29) we get

$$\left(\frac{l_{n+j}l_{\nu(r,D^n_{l,[R]}f*D^n_{l,[R]}g)+n-1}}{l_n l_{\nu(r,D^n_{l,[R]}f*D^n_{l,[R]}g)+n+j-1}}\right)^2 \le \frac{\mu(r, D^{n+j}_{l,[R]}f * D^{n+j}_{l,[R]}g)}{\mu(r, D^n_{l,[R]}f * D^n_{l,[R]}g)} \le \left(\frac{l_{n+j}l_{\nu(r,D^{n+j}_{l,[R]}f*D^{n+j}_{l,[R]}g)+n-1}}{l_n l_{\nu(r,D^{n+j}_{l,[R]}f*D^{n+j}_{l,[R]}g)+n+j-1}}\right)^2.$$

whence in view of (17) as usual we get

$$(1+o(1))\frac{l^2_{n+j}}{l^2_n}(q\nu(r, D^n_{l,[R]}f * D^n_{l,[R]}g)^{2j} \le \frac{\mu(r, D^{n+j}_{l,[R]}f * D^{n+j}_{l,[R]}g)}{\mu(r, D^n_{l,[R]}f * D^n_{l,[R]}g)} \le$$
$$\le (1+o(1))\frac{l^2_{n+j}}{l^2_n}(q\nu(r, D^{n+j}_{l,[R]}f * D^{n+j}_{l,[R]}g)^{2j}, \quad r\uparrow 1,$$

i. e.

$$q_1\nu(r, D^n_{l,[R]}f * D^n_{l,[R]}g) \le \sqrt[2j]{\frac{\mu(r, D^{n+j}_{l,[R]}f * D^{n+j}_{l,[R]}g)}{\mu(r, D^n_{l,[R]}f * D^n_{l,[R]}g)}} \le Q_1\nu(r, D^{n+j}_{l,[R]}f * D^{n+j}_{l,[R]}g)$$

for some $0 < q_1 \le Q_1 < +\infty$ and $r \in [r_0, 1)$. The further proof of Theorem 7 is the same as the proof of Theorem 6. □

We also note that by the usual method it is not difficult to prove the following statement.

Proposition 7. *Let $n \in \mathbb{N}$, $j \in \mathbb{N}$ and (17) hold. Then*

$$2j\max\{\varrho^{(1)}[f], \varrho^{(1)}[g])\} \le \varlimsup_{r\uparrow 1}\frac{1}{\ln(1/(1-r))}\ln\frac{\mu(r, D^{n+j}_{l,[R]}f * D^{n+j}_{l,[R]}g)}{\mu(r, D^n_{l,[R]}f * D^n_{l,[R]}g)} \le$$
$$\le 2j(\max\{\varrho^{(1)}[f], \varrho^{(1)}[g])\} + 1).$$

and if, moreover, $|f_k/f_{k+1}| \nearrow 1$, $|g_k/g_{k+1}| \nearrow 1$ and $l_{k-1}l_{k+1}/l_k^2 \nearrow 1$ as $k_0 \le k \to \infty$ then

$$2j\max\{\lambda^{(1)}[f], \lambda^{(1)}[g])\} \le \varliminf_{r\uparrow 1}\frac{1}{\ln(1/(1-r))}\ln\frac{\mu(r, D^{n+j}_{l,[R]}f * D^{n+j}_{l,[R]}g)}{\mu(r, D^n_{l,[R]}f * D^n_{l,[R]}g)} \le$$
$$\le 2j(\min\{\max\{\lambda^{(1)}[f], \varrho^{(1)}[g]\}, \max\{\lambda^{(1)}[g], \varrho^{(1)}[f]\}\} + 1).$$

Finally, from (31) we obtain

$$\frac{l^2_{n+j}l_{\nu(r,D^n_{l,[R]}(f*g))+n-1}l_{\nu(r,D^n_{l,[R]}(f*g))-1}}{l_n l^2_{\nu(r,D^n_{l,[R]}(f*g))+n+j-1}} \le \frac{\mu(r, D^{n+j}_{l,[R]}f * D^{n+j}_{l,[R]}g)}{\mu(r, D^n_{l,[R]}(f*g))} \le$$
$$\le \frac{l^2_{n+j}l_{\nu(r,D^{n+j}_{l,[R]}f*D^{n+j}_{l,[R]}g)-1}l_{\nu(r,D^{n+j}_{l,[R]}f*D^{n+j}_{l,[R]}g)+n-1}}{l_n l^2_{\nu(r,D^{n+j}_{l,[R]}f*D^{n+j}_{l,[R]}g)+n+j-1}},$$

whence in view of (17)

$$(1+o(1))\frac{l^2_{n+j}}{l_n}(q\nu(r, D^n_{l,[R]}(f*g)))^{n+2j} \le \frac{\mu(r, D^{n+j}_{l,[R]}f * D^{n+j}_{l,[R]}g)}{\mu(r, D^n_{l,[R]}(f*g))} \le$$

$$\le \frac{l_{n+j}^2}{l_n}(Q\nu(r, D_{l,[R]}^{n+j}f * D_{l,[R]}^{n+j}g)^{n+2j}, \quad r \uparrow 1,$$

i. e.

$$q_1\nu(r, D_{l,[R]}^n(f*g)) \le \sqrt[n+2j]{\frac{\mu(r, D_{l,[R]}^{n+j}f * D_{l,[R]}^{n+j}g)}{\mu(r, D_{l,[R]}^n(f*g))}} \le\le Q_1\nu(r, D_{l,[R]}^{n+j}f * D_{l,[R]}^{n+j}g)$$

for some $0 < q_1 \le Q_1 < +\infty$ and $r \in [r_0, 1)$.

Therefore, using the applied methodology above, we easily arrive at the correctness of the following two statements.

Theorem 8. *Let the functions α and β satisfy the assumptions of Theorem 2, $n \in \mathbb{N}$ and $j \in \mathbb{N}$ and (17) hold. Then*

$$\overline{\lim_{r\uparrow 1}} \frac{1}{\beta(1/(1-r))}\alpha\left(\sqrt[n+2j]{\frac{\mu(r, D_{l,[R]}^{n+j}f * D_{l,[R]}^{n+j}g)}{\mu(r, D_{l,[R]}^n(f*g))}}\right) = \max\{\varrho_{\alpha,\beta}^{(1)}[f], \varrho_{\alpha,\beta}^{(1)}[g]\}$$

and if, moreover, $|f_k/f_{k+1}| \nearrow 1$, $|g_k/g_{k+1}| \nearrow 1$ and $l_{k-1}l_{k+1}/l_k^2 \nearrow 1$ as $k_0 \le k \to \infty$ then

$$\max\{\lambda_{\alpha,\beta}^{(1)}[f], \lambda_{\alpha,\beta}^{(1)}[g]\} \le \overline{\lim_{r\uparrow 1}} \frac{1}{\beta(1/(1-r))}\alpha\left(\sqrt[n+2j]{\frac{\mu(r, D_{l,[R]}^{n+j}f * D_{l,[R]}^{n+j}g)}{\mu(r, D_{l,[R]}^n(f*g))}}\right) \le$$
$$\le \min\{\max\{\lambda_{\alpha,\beta}^{(1)}[f], \varrho_{\alpha,\beta}^{(1)}[g]\}, \max\{\lambda_{\alpha,\beta}^{(1)}[g], \varrho_{\alpha,\beta}^{(1)}[f]\}\}.$$

Proposition 8. *Let $n \in \mathbb{N}$, $j \in N$ and (17) hold. Then*

$$(n+2j)\max\{\varrho^{(1)}[f], \varrho^{(1)}[g])\} \le \overline{\lim_{r\uparrow 1}} \frac{1}{\ln(1/(1-r))}\ln\frac{\mu(r, D_{l,[R]}^{n+j}f * D_{l,[R]}^{n+j}g)}{\mu(r, D_{l,[R]}^n(f*g))} \le$$
$$\le (n+2j)(\max\{\varrho^{(1)}[f], \varrho^{(1)}[g])\} + 1).$$

and if, moreover, $|f_k/f_{k+1}| \nearrow 1$, $|g_k/g_{k+1}| \nearrow 1$ and $l_{k-1}l_{k+1}/l_k^2 \nearrow 1$ as $k_0 \le k \to \infty$ then

$$(n+2j)\max\{\lambda^{(1)}[f], \lambda^{(1)}[g])\} \le \underline{\lim_{r\uparrow 1}} \frac{1}{\ln(1/(1-r))}\ln\frac{\mu(r, D_{l,[R]}^{n+j}f * D_{l,[R]}^{n+j}g)}{\mu(r, D_{l,[R]}^n(f*g))} \le$$
$$\le (n+2j)(\min\{\max\{\lambda^{(1)}[f], \varrho^{(1)}[g]\}, \max\{\lambda^{(1)}[g], \varrho^{(1)}[f]\}\} + 1).$$

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Ivan Franko National University of Lviv
Lviv, Ukraine
m.m.sheremeta@gmail.com

Received 02.06.2020
Revised 02.09.2020