

УДК 517.53

S. G. KIM

**THE NORMING SET OF A SYMMETRIC BILINEAR FORM  
ON THE PLANE WITH THE SUPREMUM NORM**

S. G. Kim. *The norming set of a symmetric bilinear form on the plane with the supremum norm*, Mat. Stud. **55** (2021), 171–180.

An element  $(x_1, \dots, x_n) \in E^n$  is called a *norming point* of  $T \in \mathcal{L}_s(^nE)$  if  $\|x_1\| = \dots = \|x_n\| = 1$  and  $|T(x_1, \dots, x_n)| = \|T\|$ , where  $\mathcal{L}_s(^nE)$  denotes the space of all symmetric continuous  $n$ -linear forms on  $E$ . For  $T \in \mathcal{L}_s(^nE)$ , we define

$$\text{Norm}(T) = \{(x_1, \dots, x_n) \in E^n : (x_1, \dots, x_n) \text{ is a norming point of } T\}.$$

$\text{Norm}(T)$  is called the *norming set* of  $T$ . We classify  $\text{Norm}(T)$  for every  $T \in \mathcal{L}_s(^2l_\infty^2)$ .

**1. Introduction.** Let  $n \in \mathbb{N}, n \geq 2$ . We write  $S_E$  for the unit sphere of a Banach space  $E$ . We denote by  $\mathcal{L}(^nE)$  the Banach space of all continuous  $n$ -linear forms on  $E$  endowed with the norm  $\|T\| = \sup_{(x_1, \dots, x_n) \in S_E \times \dots \times S_E} |T(x_1, \dots, x_n)|$ .  $\mathcal{L}_s(^nE)$  denotes the closed subspace of all continuous symmetric  $n$ -linear forms on  $E$ . An element  $(x_1, \dots, x_n) \in E^n$  is called a *norming point* of  $T$  if  $\|x_1\| = \dots = \|x_n\| = 1$  and  $|T(x_1, \dots, x_n)| = \|T\|$ .

For  $T \in \mathcal{L}(^nE)$ , we define

$$\text{Norm}(T) = \{(x_1, \dots, x_n) \in E^n : (x_1, \dots, x_n) \text{ is a norming point of } T\}.$$

$\text{Norm}(T)$  is called the *norming set* of  $T$ . Notice that  $(x_1, \dots, x_n) \in \text{Norm}(T)$  if and only if  $(\epsilon_1 x_1, \dots, \epsilon_n x_n) \in \text{Norm}(T)$  for some  $\epsilon_k = \pm 1$  ( $k = 1, \dots, n$ ). Indeed, if  $(x_1, \dots, x_n) \in \text{Norm}(T)$ , then

$$|T(\epsilon_1 x_1, \dots, \epsilon_n x_n)| = |\epsilon_1 \dots \epsilon_n T(x_1, \dots, x_n)| = |T(x_1, \dots, x_n)| = \|T\|,$$

which shows that  $(\epsilon_1 x_1, \dots, \epsilon_n x_n) \in \text{Norm}(T)$ . If  $(\epsilon_1 x_1, \dots, \epsilon_n x_n) \in \text{Norm}(T)$  for some  $\epsilon_k = \pm 1$  ( $k = 1, \dots, n$ ), then

$$(x_1, \dots, x_n) = (\epsilon_1(\epsilon_1 x_1), \dots, \epsilon_n(\epsilon_n x_n)) \in \text{Norm}(T).$$

The following examples show that  $\text{Norm}(T)$  is either equal to the empty set or it is an infinite set.

Examples. (a) Let

$$T((x_i)_{i \in \mathbb{N}}, (y_i)_{i \in \mathbb{N}}) = \sum_{i, j \in \mathbb{N}} a_{ij} x_i y_j \in \mathcal{L}_s(^2c_0)$$

2010 *Mathematics Subject Classification*: 46A22.

*Keywords*: norming points; symmetric bilinear forms.

doi:10.30970/ms.55.2.171-180

for some  $a_{ij} \in \mathbb{R}$  such that  $\|T\| = \sum_{i,j \in \mathbb{N}} |a_{ij}|$ . We claim that if  $A = \{(i, j) \in \mathbb{R}^2 : a_{ij} \neq 0\}$  is an infinite set, then  $\text{Norm}(T) = \emptyset$ .

Assume that  $\text{Norm}(T) \neq \emptyset$ . Let  $((x_i)_{i \in \mathbb{N}}, (y_i)_{i \in \mathbb{N}}) \in \text{Norm}(T)$ . Then,

$$\|T\| = |T((x_i)_{i \in \mathbb{N}}, (y_i)_{i \in \mathbb{N}})| \leq \sum_{(i,j) \in A} |a_{ij}| |x_i y_j| \leq \sum_{(i,j) \in A} |a_{ij}| = \|T\|,$$

which shows that  $|x_i y_j| = 1$  for all  $(i, j) \in A$ . Hence,  $|x_i| = 1$  for infinitely many  $i \in \mathbb{N}$  or  $|y_j| = 1$  for infinitely many  $j \in \mathbb{N}$  since  $A$  is an infinite set. Therefore,  $(x_i)_{i \in \mathbb{N}} \notin c_0$  or  $(y_i)_{i \in \mathbb{N}} \notin c_0$ . This is a contradiction. Therefore,  $\text{Norm}(T) = \emptyset$ .

From this, if

$$T((x_i)_{i \in \mathbb{N}}, (y_i)_{i \in \mathbb{N}}) = \sum_{i=1}^{\infty} \frac{1}{2^i} x_i y_i \in \mathcal{L}_s(^2 c_0),$$

then  $\text{Norm}(T) = \emptyset$ .

We do not know whether there is some  $T \in \mathcal{L}_s(^2 c_0)$  such that  $\text{Norm}(T)$  is a nonempty finite set.

(b) Let

$$T((x_i)_{i \in \mathbb{N}}, (y_i)_{i \in \mathbb{N}}) = x_1 y_1 \in \mathcal{L}_s(^2 c_0).$$

Then,

$$\text{Norm}(T) = \{((\pm 1, x_2, x_3, \dots), (\pm 1, y_2, y_3, \dots)) \in c_0 \times c_0 : |x_j| \leq 1, |y_j| \leq 1 \text{ for } j \geq 2\}.$$

A mapping  $P: E \rightarrow \mathbb{R}$  is a continuous  $n$ -homogeneous polynomial if there exists a continuous  $n$ -linear form  $L$  on the product  $E \times \dots \times E$  such that  $P(x) = L(x, \dots, x)$  for every  $x \in E$ . We denote by  $\mathcal{P}(^n E)$  the Banach space of all continuous  $n$ -homogeneous polynomials from  $E$  into  $\mathbb{R}$  endowed with the norm  $\|P\| = \sup_{\|x\|=1} |P(x)|$ .

An element  $x \in E$  is called a *norming point* of  $P \in \mathcal{P}(^n E)$  if  $\|x\| = 1$  and  $|P(x)| = \|P\|$ . For  $P \in \mathcal{P}(^n E)$ , we define

$$\text{Norm}(P) = \{x \in E : x \text{ is a norming point of } P\}.$$

$\text{Norm}(P)$  is called the *norming set* of  $P$ . Notice that  $\text{Norm}(P) = \emptyset$  or a finite set or an infinite set.

Recently, Kim [8] classify  $\text{Norm}(P)$  for every  $P \in \mathcal{P}(^2 l_\infty^2)$ , where  $l_\infty^2 = \mathbb{R}^2$  with the supremum norm.

If  $\text{Norm}(T) \neq \emptyset$ ,  $T \in \mathcal{L}(^n E)$  is called a *norm attaining  $n$ -linear form* and if  $\text{Norm}(P) \neq \emptyset$ ,  $P \in \mathcal{P}(^n E)$  is called a *norm attaining  $n$ -homogeneous polynomial* (see [3]).

For more details about the theory of multilinear mappings and polynomials on a Banach space, we refer to [5].

Let us introduce a brief history of norm attaining multilinear forms and polynomials on Banach spaces. In 1961 Bishop and Phelps [2] initiated and showed that the set of norm attaining functionals on a Banach space is dense in the dual space. Shortly after, attention was paid to possible extensions of this result to more general settings, specially bounded linear operators between Banach spaces. The problem of denseness of norm attaining functions has moved to other types of mappings like multilinear forms or polynomials. The first result about norm attaining multilinear forms appeared in a joint work of Aron, Finet and Werner [1], where they showed that the Radon-Nikodym property is sufficient for the denseness

of norm attaining multilinear forms. Choi and Kim [3] showed that the Radon-Nikodym property is also sufficient for the denseness of norm attaining polynomials. Jimenez-Sevilla and Paya [6] studied the denseness of norm attaining multilinear forms and polynomials on preduals of Lorentz sequence spaces.

It seems to be natural and interesting to study about  $\text{Norm}(T)$  for  $T \in \mathcal{L}_s({}^n E)$ . For  $m \in \mathbb{N}$ , let  $l_\infty^m := \mathbb{R}^m$  with the supremum norm. Notice that for every  $T \in \mathcal{L}_s({}^n l_\infty^m)$ ,  $\text{Norm}(T) \neq \emptyset$  since  $S_{l_\infty^m}$  is compact.

In this paper, we classify  $\text{Norm}(T)$  for every  $T \in \mathcal{L}_s({}^2 l_\infty^2)$ .

## 2. Results.

**Theorem A** ([7]). *Let  $T((x_1, x_2), (y_1, y_2)) = ax_1y_1 + bx_2y_2 + c(x_1y_2 + x_2y_1) \in \mathcal{L}_s({}^2 l_\infty^2)$ . Then,*

$$\|T\| = \max\{|a + b| + 2|c|, |a - b|\}.$$

Notice that if  $\|T\| = 1$ , then  $|a| \leq 1$ ,  $|b| \leq 1$ ,  $|c| \leq \frac{1}{2}$ .

**Lemma 1.** *Let  $T((x_1, x_2), (y_1, y_2)) = ax_1y_1 + bx_2y_2 + c(x_1y_2 + x_2y_1) \in \mathcal{L}_s({}^2 l_\infty^2)$ . Then there exists (unique)  $T'((x_1, x_2), (y_1, y_2)) = a^*x_1y_1 + b^*x_2y_2 + c^*(x_1y_2 + x_2y_1) \in \mathcal{L}_s({}^2 l_\infty^2)$  such that  $a^*, b^*, c^* \in \{\pm a, \pm b, \pm c\}$  with  $a^* \geq |b^*|$  and  $c^* \geq 0$  and  $\|T\| = \|T'\|$ .*

*Proof.* If  $a < 0$ , taking  $-T$ , we assume that  $a \geq 0$ . If  $|b| > a$ , let

$$T'_1((x_1, x_2), (y_1, y_2)) := T((x_2, x_1), (y_2, y_1)) = |b|x_1y_1 + ax_2y_2 + c(x_1y_2 + x_2y_1).$$

Then,  $\|T'_1\| = \|T\|$ . If  $c < 0$ , let

$$T'_2(((x_1, x_2), (y_1, y_2))) := T'_1((-x_1, x_2), (-y_1, y_2)) = |b|x_1y_1 + ax_2y_2 + |c|(x_1y_2 + x_2y_1).$$

Then,  $\|T'_2\| = \|T\|$ . Therefore, we can find a bilinear form  $T'$  which satisfies the conditions of the theorem.  $\square$

**Lemma 2.** *Let  $T((x_1, x_2), (y_1, y_2)) = ax_1y_1 + bx_2y_2 + c(x_1y_2 + x_2y_1) \in \mathcal{L}_s({}^2 l_\infty^2)$ . Then,  $((x_1, x_2), (y_1, y_2)) \in \text{Norm}(T)$  if and only if  $((y_1, y_2), (x_1, x_2)), (-x_1, x_2), (y_1, y_2)), ((x_1, x_2), -(y_1, y_2)), (-x_1, x_2), -(y_1, y_2)) \in \text{Norm}(T)$ .*

Let  $T((x_1, x_2), (y_1, y_2)) = ax_1y_1 + bx_2y_2 + c(x_1y_2 + x_2y_1) \in \mathcal{L}_s({}^2 l_\infty^2)$ . By Lemma 1, without loss of generality we may assume that  $a \geq |b|$  and  $c \geq 0$ .

We are in position to prove the main result of this paper.

**Theorem 1.** *Let  $T((x_1, x_2), (y_1, y_2)) = ax_1y_1 + bx_2y_2 + c(x_1y_2 + x_2y_1) \in \mathcal{L}_s({}^2 l_\infty^2)$  be such that  $\|T\| = 1$  with  $a \geq |b|$  and  $c \geq 0$ . Then,*

**Case 1:**  $b \geq 0$ .

If  $c = 0 = b$ , then

$$\text{Norm}(T) = \{(\pm(1, t), \pm(1, s)) : -1 \leq t, s \leq 1\}.$$

If  $c = 0, b > 0$ , then

$$\text{Norm}(T) = \{(\pm(1, 1), \pm(1, 1)), (\pm(1, -1), \pm(1, -1))\}.$$

If  $c > 0$ , then

$$\text{Norm}(T) = \{(\pm(1, 1), \pm(1, 1))\}.$$

**Case 2:**  $b < 0$

**Subcase 1:**  $1 = |a + b| + 2|c| > |a - b|$ .

If  $c < \frac{1}{2}$ , then

$$\text{Norm}(T) = \{(\pm(1, 1), \pm(1, 1))\}.$$

If  $c = \frac{1}{2}$ , then

$$\text{Norm}(T) = \{(\pm(1, 1), \pm(1, 1)), (\pm(1, -1), \pm(1, -1))\}.$$

**Subcase 2:**  $|a + b| + 2|c| = |a - b| = 1$ .

If  $a = \frac{1}{2}$ , then

$$\begin{aligned} \text{Norm}(T) = & \{(\pm(1, -1), \pm(s, 1)), (\pm(s, 1), \pm(1, -1)), (\pm(1, t), \pm(1, 1)), \\ & (\pm(1, 1), \pm(1, t)) : -1 \leq s, t \leq 1\}. \end{aligned}$$

If  $\frac{1}{2} < a$ , then

$$\begin{aligned} \text{Norm}(T) = & \{(\pm(1, t), \pm(1, 1)), (\pm(1, 1), \pm(1, t)), (\pm(1, 1), \pm(1, -1)), \\ & (\pm(1, -1), \pm(1, 1)) : -1 \leq t \leq 1\}. \end{aligned}$$

**Subcase 3:**  $|a + b| + 2|c| < |a - b| = 1$ .

$$\text{Norm}(T) = \{(\pm(1, -1), \pm(1, 1)), (\pm(1, 1), \pm(1, -1))\}.$$

*Proof.* Let  $((x_1, x_2), (y_1, y_2)) \in \text{Norm}(T)$ . By Lemma 2, we may assume that  $(x_1 = y_1 = 1)$  or  $(x_1 = y_2 = 1)$  or  $(x_2 = y_2 = 1)$ .

**Case 1:**  $b \geq 0$ .

By Theorem A,

$$\|T\| = 1 = |a + b| + 2|c| = a + b + 2c > |a - b| = a - |b|.$$

Let  $c > 0$ . Suppose that  $((1, t), (1, y)) \in \text{Norm}(T)$  for some  $t, y \in [-1, 1]$ . Notice that if  $c = 0 = b$ , then  $a = 1$  and  $T((x_1, x_2), (y_1, y_2)) = x_1 y_1$ . Hence,

$$\text{Norm}(T) = \{(\pm(1, t), \pm(1, s)) : -1 \leq t, s \leq 1\}.$$

Let  $c = 0$  and  $b > 0$ . Suppose that  $((1, t), (1, y)) \in \text{Norm}(T)$  for some  $t, y \in [-1, 1]$ . It follows that

$$1 = |T((1, t), (1, y))| \leq |a + bty| \leq a + b|t||y| \leq a + b = 1,$$

which implies that  $1 = ty$ . Hence,  $((1, 1), (1, 1)), ((1, -1), (1, -1)) \in \text{Norm}(T)$ .

Suppose that  $((1, t), (y, 1)) \in \text{Norm}(T)$  for some  $t, y \in [-1, 1]$ . It follows that

$$1 = |T((1, t), (y, 1))| \leq |ay + bt| \leq a|y| + b|t| \leq a + b = 1,$$

which implies that  $t = y = \pm 1$ . Hence,  $((1, 1), (1, 1)), ((1, -1), (-1, 1)) \in \text{Norm}(T)$ .

Suppose that  $((t, 1), (y, 1)) \in \text{Norm}(T)$  for some  $t, y \in [-1, 1]$ . It follows that

$$1 = |T((t, 1), (y, 1))| \leq |aty + b| \leq a|t||y| + b \leq a + b = 1,$$

which implies that  $1 = ty$ . Hence,  $((1, 1), (1, 1)), ((-1, 1), (-1, 1)) \in \text{Norm}(T)$ . By Lemma 2,

$$\text{Norm}(T) = \{(\pm(1, 1), \pm(1, 1)), (\pm(1, -1), \pm(1, -1))\}.$$

It follows that

$$1 = |T((1, t), (1, y))| \leq |a + bty + c(t + y)| \leq a + b|t||y| + c(|t| + |y|) \leq a + b + 2c = 1,$$

which implies that  $1 = ty = t = y$ . Hence,  $((1, 1), (1, 1)) \in \text{Norm}(T)$ .

Suppose that  $((1, t), (y, 1)) \in \text{Norm}(T)$  for some  $t, y \in [-1, 1]$ . It follows that

$$1 = |T((1, t), (y, 1))| \leq |ay + bt + c(1 + ty)| \leq a|y| + b|t| + c(1 + |t||y|) \leq a + b + 2c = 1,$$

which implies that  $1 = ty = t = y$ . Hence,  $((1, 1), (1, 1)) \in \text{Norm}(T)$ .

Suppose that  $((t, 1), (y, 1)) \in \text{Norm}(T)$  for some  $t, y \in [-1, 1]$ . It follows that

$$1 = |T((t, 1), (y, 1))| \leq |aty + b + c(1 + ty)| \leq a|t||y| + b + c(1 + |t||y|) \leq a + b + 2c = 1,$$

which implies that  $1 = ty = t = y$ . Hence,  $((1, 1), (1, 1)) \in \text{Norm}(T)$ . By Lemma 2,

$$\text{Norm}(T) = \{(\pm(1, 1), \pm(1, 1))\}.$$

**Case 2:**  $b < 0$

**Subcase 1:**  $1 = |a + b| + 2|c| > |a - b|$ .

By Theorem A,

$$\|T\| = 1 = |a + b| + 2|c| = a + b + 2c > |a - b| = a - b.$$

Notice that  $c > |b| > 0$  and  $a \geq |b| > 0$ .

Let  $c < \frac{1}{2}$ . Suppose that  $((1, t), (1, y)) \in \text{Norm}(T)$  for some  $t, y \in [-1, 1]$ . Then,

$$1 = |T((1, t), (1, y))| = |a + bty + c(t + y)| = |y(c - |b|t) + a + ct|.$$

Notice that  $c - |b|t \neq 0$  because  $c > |b|$ . Suppose that  $1 = T((1, t), (1, y))$ . Then

$$y = \frac{1 - a - ct}{c - |b|t}, \quad 1 = \frac{1 - a - c}{c - |b|} \leq t.$$

Hence,  $t = 1 = y$  and  $((1, 1), (1, 1)) \in \text{Norm}(T)$ . Suppose that  $-1 = T((1, t), (1, y))$ . Then,

$$y = \frac{-1 - a - ct}{c - |b|t}, \quad t \leq \frac{-1 - a + c}{c + |b|} = -1.$$

Hence,

$$t = -1, \quad y = \frac{-1 - a + c}{c + |b|} < -1,$$

a contradiction. Hence, there are no norming points of  $T$  in this case.

Suppose that  $((1, t), (y, 1)) \in \text{Norm}(T)$  for some  $t, y \in [-1, 1]$ . Then,

$$1 = |T((1, t), (y, 1))| = |ay + bt + c(1 + ty)| = |y(a + ct) + c + bt|.$$

We claim that  $a + ct \neq 0$  for all  $t \in [-1, 1]$ . Otherwise, there is  $t \in [-1, 1]$  such that  $a + ct = 0$ . Then,

$$\pm 1 = c + bt = c + b\left(\frac{-a}{c}\right) = c + \frac{a|b|}{c}.$$

Hence,  $c = |\pm c| = c^2 + a|b| = c^2 + a(a + 2c - 1)$ , so

$$0 = c^2 + (2a - 1)c + a(a - 1) = (c + a)(c + a - 1),$$

which shows that  $c = 1 - a$ . It follows that  $1 = a + b + 2c = a + b + 2(1 - a) = 2 - a + b$ , so  $a - b = 1$ , a contradiction.

Suppose that  $1 = T((1, t), (y, 1))$ . Then,  $y = \frac{1-c-bt}{a+ct}$ . If  $a + ct > 0$ , then  $1 - c - bt \leq a + ct$ , which implies that  $t \geq \frac{1-a-c}{b+c} = 1$ . Hence,  $t = 1 = y$ . Hence,  $((1, 1), (1, 1)) \in \text{Norm}(T)$ . Let  $a + ct < 0$ . Then,  $-1 < t < 0$  and  $1 - c - bt \leq -a - ct$ . It follows that

$$2a + b + c = 1 + a - c \leq (b - c)t \leq |(b - c)t| < |b - c| = |b| + c,$$

which implies that  $a < |b|$ , a contradiction. Hence, there are no norming points of  $T$  in this case. Suppose that  $-1 = T((1, t), (y, 1))$ . Then,  $y = \frac{-1-c-bt}{a+ct}$ . If  $a + ct > 0$ , then  $-1 - c - bt \leq -1 - c - bt$ , which implies that  $t \geq \frac{1-a+c}{-b+c} > 1$ , a contradiction. Hence, there are no norming points of  $T$  in this case. Let  $a + ct < 0$ . Then,  $-1 < t < 0$  and  $-1 - c - bt \geq a + ct$ . It follows that  $-(b + c) = -|b + c| < (b + c)t < -1 - a - c$ , which implies that  $1 < b - a < 0$ , a contradiction. Hence, there are no norming points of  $T$  in this case.

Suppose that  $((t, 1), (y, 1)) \in \text{Norm}(T)$  for some  $t, y \in [-1, 1]$ . Then,

$$1 = |T((t, 1), (y, 1))| = |aty + b + c(t + y)| = |y(c + at) + b + ct|.$$

We claim that  $c + at \neq 0$  for all  $t \in [-1, 1]$ . Otherwise, there is  $t \in [-1, 1]$  such that  $c + at = 0$ . Then,

$$1 \geq |t| = \frac{c}{a}, \quad \pm 1 = b + ct = b + c\left(\frac{-c}{a}\right),$$

so  $a^2 \geq c^2 = a(b \pm 1)$ , which implies that  $a \geq \max\{b + 1, b - 1\} = b + 1$ , a contradiction since  $a - b < 1$ . Suppose that  $1 = T((t, 1), (y, 1))$ . Then,  $y = \frac{1-b-ct}{c+at}$ . If  $c + at > 0$ , then  $1 - b - ct \leq c + at$ , which implies that  $t \geq \frac{1-b-c}{a+c} = 1$ . Hence,  $t = 1 = y$  and  $((1, 1), (1, 1)) \in \text{Norm}(T)$ . If  $c + at < 0$ , then  $c + at \leq 1 - b - ct$ , which implies that  $t \leq \frac{1-b-c}{a+c} = 1$ . Thus,  $t = 1$  and  $0 < c + a = c + at < 0$ , a contradiction. Hence, there are no norming points of  $T$  in this case.

Suppose that  $-1 = T((t, 1), (y, 1))$ . Then,  $y = \frac{-1-b-ct}{c+at}$ . If  $c + at > 0$ , then  $-1 - b - ct \leq -1 - b - ct$ , which implies that  $(c - a)t \leq -1 - b + c$ . It follows that  $-|c - a| \leq (c - a)t \leq -1 - b + c$ , so  $a \leq |b|, c = \frac{1}{2}$ , a contradiction. Hence, there are no norming points of  $T$  in this case. If  $c + at < 0$ , then  $c + at \leq -1 - b - ct$ , which implies that  $t \leq \frac{-1-b-c}{a+c} < -1$ , a contradiction. Thus, there are no norming points of  $T$  in this case. Therefore,  $\text{Norm}(T) = \{(\pm(1, 1), \pm(1, 1))\}$ . Let  $c = \frac{1}{2}$ . Then,

$$T((x_1, x_2), (y_1, y_2)) = ax_1y_1 - ax_2y_2 + \frac{1}{2}(x_1y_2 + x_2y_1)$$

for some  $0 < a < \frac{1}{2}$ . We will show that

$$\text{Norm}(T) = \{(\pm(1, 1), \pm(1, 1)), (\pm(1, -1), \pm(1, -1))\}.$$

Suppose that  $((1, t), (1, y)) \in \text{Norm}(T)$  for some  $t, y \in [-1, 1]$ . Then,

$$1 = |T((1, t), (1, y))| = \left| y\left(\frac{1}{2} - at\right) + a + \frac{1}{2}t \right|.$$

Hence,

$$y = \frac{\pm 1 - a - \frac{1}{2}t}{\frac{1}{2} - at}.$$

Notice that if  $1 = T((1, t), (1, y))$ , then  $t = 1 = y$  and that if  $-1 = T((1, t), (1, y))$ , then  $t = -1 = y$ . Hence,

$$((1, 1), (1, 1)), ((1, -1), (1, -1)) \in \text{Norm}(T).$$

Suppose that  $((1, t), (y, 1)) \in \text{Norm}(T)$  for some  $t, y \in [-1, 1]$ . Then,

$$1 = |T((1, t), (y, 1))| = \left| y \left( a + \frac{1}{2}t \right) + \frac{1}{2} - at \right|.$$

Observe that  $a + \frac{1}{2}t \neq 0$  for all  $t \in [-1, 1]$ . Otherwise,  $a + \frac{1}{2}t = 0$  for some  $t \in [-1, 1]$ . Then,  $\pm 1 = \frac{1}{2} + 2a^2 < 1$ , a contradiction. Therefore,

$$y = \frac{\pm 1 - \frac{1}{2} + at}{a + \frac{1}{2}t}.$$

Suppose that  $1 = T((1, t), (y, 1))$ . If  $a + \frac{1}{2}t > 0$ , then  $t = 1 = y$  and that if  $a + \frac{1}{2}t < 0$ , then  $t = -1 = y$ . Hence,

$$((1, 1), (1, 1)), ((1, -1), (1, -1)) \in \text{Norm}(T).$$

Notice that if  $-1 = T((1, t), (y, 1))$ , there are no norming points of  $T$ .

Suppose that  $((t, 1), (y, 1)) \in \text{Norm}(T)$  for some  $t, y \in [-1, 1]$ . Then,

$$1 = |T((t, 1), (y, 1))| = \left| y \left( \frac{1}{2} + at \right) - a + \frac{1}{2}t \right|.$$

Hence,

$$y = \frac{\pm 1 + a - \frac{1}{2}t}{\frac{1}{2} + at}.$$

Notice that if  $1 = T((t, 1), (y, 1))$ , then  $t = 1 = y$  and that if  $-1 = T((t, 1), (y, 1))$ , then  $t = -1 = y$ . Thus,

$$\text{Norm}(T) = \{(\pm(1, 1), \pm(1, 1)), (\pm(1, -1), \pm(1, -1))\}.$$

**Subcase 2:**  $|a + b| + 2|c| = |a - b| = 1$ .

Notice that  $T((x_1, x_2), (y_1, y_2)) = ax_1y_1 - (1-a)x_2y_2 + (1-a)(x_1y_2 + x_2y_1)$  with  $\frac{1}{2} \leq a < 1$ .

If  $a = \frac{1}{2}$ , then  $T((x_1, x_2), (y_1, y_2)) = \frac{1}{2}x_1y_1 - \frac{1}{2}x_2y_2 + \frac{1}{2}(x_1y_2 + x_2y_1)$ . Hence,

$$\begin{aligned} \text{Norm}(T) &= \{(\pm(1, -1), \pm(s, 1)), (\pm(s, 1), \pm(1, -1)), (\pm(1, t), \pm(1, 1)), \\ &\quad (\pm(1, 1), \pm(1, t)) : -1 \leq s, t \leq 1\}. \end{aligned}$$

Let  $\frac{1}{2} < a$ . Suppose that  $((1, t), (1, y)) \in \text{Norm}(T)$  for some  $t, y \in [-1, 1]$ . Then,

$$1 = |T((1, t), (1, y))| = |a - (1-a)ty + (1-a)(t+y)| = |y(1-a)(1-t) + a + (1-a)t|.$$

Therefore,  $y = \frac{\pm 1 - a - (1-a)t}{(1-a)(1-t)}$ . Suppose that  $1 = T((1, t), (1, y))$ . Then,  $y = 1$  for all  $t \in [-1, 1]$ .

Hence,  $((1, t), (1, 1)) \in \text{Norm}(T)$  for  $-1 \leq t \leq 1$ . Suppose that  $-1 = T((1, t), (1, y))$ . Then,

$$y = \frac{-1 - a - (1-a)t}{(1-a)(1-t)} < -1,$$

which is a contradiction. Thus, there are no norming points of  $T$  in this case.

Suppose that  $((1, t), (y, 1)) \in \text{Norm}(T)$  for some  $t, y \in [-1, 1]$ . Then,

$$1 = |T((1, t), (y, 1))| = |y(a + (1 - a)t) + (1 - a)(1 - t)|.$$

Notice that  $a + (1 - a)t \neq 0$  for all  $t \in [-1, 1]$  because  $a > \frac{1}{2}$ . Hence,  $y = \frac{\pm 1 - (1 - a)(1 - t)}{a + (1 - a)t}$ . Suppose that  $1 = T((1, t), (y, 1))$ . Then,  $y = 1$  for all  $x \in [-1, 1]$ . Thus,  $((1, t), (1, 1)) \in \text{Norm}(T)$  for  $-1 \leq t \leq 1$ .

Suppose that  $-1 = T((1, t), (y, 1))$ . Then,

$$y = 1 - \frac{2}{a + (1 - a)t} \leq 1 - \frac{2}{a + (1 - a)|t|} \leq 1 - \frac{2}{a + (1 - a)} = -1.$$

Then,  $y = -1, t = 1$ . Hence,  $((1, 1), (-1, 1)) \in \text{Norm}(T)$ .

Suppose that  $((t, 1), (y, 1)) \in \text{Norm}(T)$  for some  $t, y \in [-1, 1]$ . Then,

$$1 = |T((t, 1), (y, 1))| = |y(at + (1 - a)) - (1 - a)(1 - t)|.$$

We claim that  $at + (1 - a) \neq 0$  for all  $t \in [-1, 1]$ . Otherwise, there is  $t \in [-1, 1]$  such that  $at + (1 - a) = 0$ . Then,

$$t = \frac{-(1 - a)}{a}, \quad \pm 1 + (1 - a)\left(1 + \frac{1 - a}{a}\right) = 0,$$

which is impossible. Hence,

$$y = \frac{\pm 1 + (1 - a)(1 - t)}{(1 - a) + at}.$$

Suppose that  $1 = T((t, 1), (y, 1))$ . Then,

$$y = \frac{1 + (1 - a)(1 - t)}{at + (1 - a)}.$$

If  $at + (1 - a) > 0$ , then  $1 + (1 - a)(1 - t) \leq at + (1 - a)$ . Hence,  $t = 1 = y$ . Therefore,  $((1, 1), (1, 1)) \in \text{Norm}(T)$ . If  $at + (1 - a) < 0$ , then  $-at - (1 - a) \geq 1 + (1 - a)(1 - t)$ . Hence,  $t \leq \frac{3 - 2a}{1 - 2a} < -1$  a contradiction. Thus, there are no norming points of  $T$  in this case.

Suppose that  $-1 = T((t, 1), (y, 1))$ . Then,

$$y = \frac{at - (a + t)}{at - (a - 1)} \geq 1,$$

so  $y = 1, t = -1$ . Hence,  $((-1, 1), (1, 1)) \in \text{Norm}(T)$ . Therefore,

$$\begin{aligned} \text{Norm}(T) = & \{(\pm(1, t), \pm(1, 1)), (\pm(1, 1), \pm(1, t)), (\pm(1, 1), \pm(1, -1)), \\ & (\pm(1, -1), \pm(1, 1)) : -1 \leq t \leq 1\}. \end{aligned}$$

**Subcase 3:**  $|a + b| + 2|c| < |a - b| = 1$ .

By Theorem A,

$$|a + b| + 2|c| = a - |b| + 2c < |a - b| = a + |b| = 1 = \|T\|.$$

Notice that  $a \geq |b| > c$ . Suppose that  $((1, t), (1, y)) \in \text{Norm}(T)$  for some  $t, y \in [-1, 1]$ . Then,

$$1 = |T((1, t), (1, y))| = |a + bty + c(t + y)| = |y(c - |b|t) + a + ct|.$$



We claim that  $c - |b|t \neq 0$ . Otherwise, there is  $t \in [-1, 1]$  such that  $c - |b|t = 0$ . Then,  $t = \frac{c}{|b|}$ ,  $\pm 1 - a - ct = 0$ . Hence,

$$0 = \left( \pm 1 - a - c \left( \frac{c}{|b|} \right) \right) |b|,$$

so  $0 = |b|^2 - c^2$  or  $-(1+a)|b| = c^2$ , which imply that  $|b| = c$  or  $c^2 < 0$ . These are contradictions.

Suppose that  $1 = T((1, t), (1, y)) = a + bty + c(t + y)$ . If  $c - |b|t > 0$ , then

$$y = \frac{1 - a - ct}{c - |b|t}, \quad t \leq \min \left\{ \frac{1 - a - c}{c - |b|}, \frac{1 - a + c}{c + |b|} \right\} = \min\{-1, 1\} = -1.$$

Hence,  $t = -1, y = 1$ , and  $((1, -1), (1, 1)) \in \text{Norm}(T)$ . If  $c - |b|t < 0$ , then

$$y = \frac{1 - a - ct}{c - |b|t}, \quad 1 = \frac{1 - a + c}{c + |b|} \leq t \leq \frac{1 - a - c}{c - |b|} = -1,$$

a contradiction. Hence, there are no norming points of  $T$  in this case.

Suppose that  $-1 = T((1, t), (1, y))$ . If  $c - |b|t > 0$ , then

$$y = \frac{-1 - a - ct}{c - |b|t}, \quad t \leq \min \left\{ \frac{-1 - a - c}{c - |b|}, \frac{-1 - a + c}{c + |b|} \right\} = \frac{-1 - a + c}{c + |b|} < -1,$$

a contradiction. Hence, there are no norming points of  $T$  in this case. If  $c - |b|t < 0$ , then  $y = \frac{-1 - a - ct}{c - |b|t} > 1$  because  $(|b| - c)t \leq |b| - c \leq |b| < 1 + a + c$ . This is a contradiction. Therefore, there are no norming points of  $T$  in this case.

Suppose that  $((1, t), (y, 1)) \in \text{Norm}(T)$  for some  $t, y \in [-1, 1]$ . Then,

$$1 = |T((1, t), (y, 1))| = |ay + bt + c(1 + ty)| = |y(a + ct) + c + bt|.$$

Since  $a > c$ ,  $a + ct \neq 0$ . Suppose that  $1 = T((1, t), (y, 1))$ . Then,

$$y = \frac{1 - c - bt}{a + ct}, \quad t \leq \frac{1 - a - c}{c - |b|} = -1.$$

Hence,  $t = -1, y = 1$ , and  $((1, -1), \pm(1, 1)), (-1, 1), \pm(1, -1)) \in \text{Norm}(T)$ .

Suppose that  $-1 = T((1, t), (y, 1))$ . Then,

$$y = \frac{-1 - c - bt}{a + ct}, \quad 1 = \frac{-1 - a + c}{c + |b|} \leq t.$$

Therefore,  $t = 1, y = -1$ , and  $((1, 1), (-1, 1)) \in \text{Norm}(T)$ . We claim that  $c + at \neq 0$  for all  $t \in [-1, 1]$ . Indeed, assume that  $c + at = 0$  for some  $t \in [-1, 1]$ . Then,  $\pm 1 = b + ct = b - c \left( \frac{c}{a} \right)$ , which implies that  $\pm a = ab - c^2$ . Hence,  $a = |\pm a| = |ab - c^2| = a|b| + c^2$ , so  $a^2 = a(1 - |b|) = c^2$ , so  $a = c$ , which is impossible since  $a > c$ . Suppose that  $((t, 1), (y, 1)) \in \text{Norm}(T)$  for some  $t, y \in [-1, 1]$ . Then,

$$1 = |T((t, 1), (y, 1))| = |aty + b + c(t + y)| = |y(c + at) + b + ct|.$$

Suppose that  $1 = T((t, 1), (y, 1)) = aty + b + c(t + y)$ . Let  $c > 0$ . Then,  $y = \frac{1-b-ct}{c+at}$ . Note that if  $c + at > 0$ , then  $t \geq \frac{1-b-c}{a+c} > 1$ . Hence, there are no norming points of  $T$  in this case. Let  $c + at < 0$ . Then,  $t \leq \frac{1-b+c}{c-a} < -1$ . Thus, there are no norming points of  $T$  in this case.

Suppose that  $-1 = T((t, 1), (y, 1))$ . Then,  $y = \frac{-1-b-ct}{c+at}$ . If  $c + at > 0$ , then

$$t \geq \max \left\{ \frac{1+b-c}{a-c}, \frac{-1-b-c}{a+c} \right\} = \max\{1, -1\} = 1.$$

Hence,  $t = 1, y = -1$ , and  $((1, 1), (-1, 1)) \in \text{Norm}(T)$ . If  $c + at < 0$ , then

$$t \leq \min \left\{ \frac{1+b-c}{a-c}, \frac{-1-b-c}{a+c} \right\} = \min\{1, -1\} = -1.$$

Therefore,  $t = -1, y = 1$ , and  $((-1, 1), (1, 1)) \in \text{Norm}(T)$ . By Lemma 2,

$$\text{Norm}(T) = \{(\pm(1, -1), \pm(1, 1)), (\pm(1, 1), \pm(1, -1))\}.$$

□

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Department of Mathematics, Kyungpook National University,  
Daegu 702-701, Republic of Korea  
sgk317@knu.ac.kr

Received 07.12.2020

Revised 20.01.2021