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LOGARITHMIC DERIVATIVE ESTIMATES OF MEROMORPHIC FUNCTIONS OF FINITE ORDER IN THE HALF-PLANE

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We established new sharp estimates outside exceptional sets for the logarithmic derivatives $\frac{d^k \log f(z)}{dz^k}$ and its generalizations $\frac{f^{(k)}(z)}{f^{(j)}(z)}$, where f is a meromorphic function f in the upper half-plane, $k > j \geq 0$ are integers. These estimates improve known estimates due to the second author in the class of meromorphic functions of finite order. Examples show that size of exceptional sets are best possible in some sense.

Estimates of the logarithmic derivative $\frac{f'}{f}$ and its generalizations have many applications in different branches of analysis and differential equations [12]–[13]. They are of particular importance in the Nevanlinna theory, where the Lemma of the logarithmic derivative plays an important role (see [2], [10], [11], [12], [15], [17],).

While in the mentioned results the integral means $\log\left|\frac{f'}{f}\right|$ is estimated, in differential equations one needs estimates of the mean values of $\left|\frac{f'}{f}\right|$ and uniform estimates of the logarithmic derivatives. The case of meromorphic or entire function f is well studied. For instance, G. Gundersen [13] proved that for a meromorphic in $\mathbb{C}$ function f of finite order ρ one has

$$|f'(z)/f(z)| < K|z|^{\rho+\varepsilon}, \quad z \in \mathbb{C} \setminus E, \tag{1}$$

where E is the set of disks with finite sum of radii. This result improved an old result of G. Valiron ([19, p. 87]). Recent sharp uniform estimates of the logarithmic derivative (see e.g. [3], [4]) are based on deep results due to J. Anderson and V. Eiderman. [1].

A counterpart of (1) for meromorphic functions in the unit disk was obtained in [5]. Similar sharp estimates with applications to complex differential equations can be found in [6], [7]. One of the most general estimates covering both complex plane and unit disk cases have been recently obtained in [8].

Let

$$\mathbb{C}_+(r_0) = \{z = re^{i\theta} : 0 \leqslant \theta \leqslant \pi, r_0 \leqslant r < +\infty\},$$

$w(z)$, $z \in \mathbb{C}_+(r_0)$ be a meromorphic function. Consider Nevanlinna's characteristics of $w(z)$, $z \in \mathbb{C}_+(r_0)$ [12, p.40]. Denote

$$\begin{gathered} A(r,w) = \frac{1}{\pi}\int_{r_0}^{r}\left(\frac{1}{t^2} - \frac{1}{r^2}\right)\left(\log^+|w(t)| + \log^+|w(-t)|\right)dt, \\ B(r,w) = \frac{2}{\pi r}\int_0^{\pi}\log^+|w(re^{i\theta})|\sin\theta\, d\theta, \\ C(r,w) = 2\int_{r_0}^{r}c(t,w)\left(\frac{1}{t^2} + \frac{1}{r^2}\right)\, dt, \end{gathered} \tag{2}$$

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where $\log^+ x = \max(\log x, 0),\ x \geqslant 0$,

$$c(t,w) = c(t,\infty) = \sum_{r_0<|b_l|\leqslant t, 0\leqslant\theta_l\leqslant\pi} \sin\theta_l,$$

is the counting function of poles, each pole $b_l = |b_l|e^{i\theta_l}$ is counted due to its multiplicity,

$$S(r,w) = A(r,w) + B(r,w) + C(r,w),\ r_0 \leqslant r < +\infty. \tag{3}$$

We denote (see (2))

$$c(t,0,\infty) = c(t,w) + c(t,1/w), \tag{4}$$

i.e. $c(t,0,\infty)$ is the counting function of zeros and poles of f.

It is well-known [12, p. 39–41], that

$$|\log|f|| = \log^+|f| + \log^+|1/f|;$$

$$\max\Big\{B(r,f) + B\Big(r,\frac{1}{f}\Big), A(r,f) + A\Big(r,\frac{1}{f}\Big), C(r,f) + C\Big(r,\frac{1}{f}\Big)\Big\} < 2S(r,f) + \text{const}. \tag{5}$$

The quantity

$$\rho[w] = \varlimsup_{r\to+\infty} \frac{\log^+ S(r,w)}{\log r} \tag{6}$$

is called the *growth order* of the function $w(z)$, $z \in \mathbb{C}_+(r_0)$.

Though the unit disk is conformally equivalent to the half-plane $\mathbb{C}_+ = \{z : \operatorname{Im} z > 0\}$, it is impossible to transfer directly the mentioned results to functions meromorphic even in $\overline{\mathbb{C}}_+$. In [14] an example is constructed showing that there is no counterpart of the logarithmic derivative lemma of Nevanlinna [12, p. 116, 137] for functions meromorphic in the upper half-plane (cf. [10]). In particular, it follows that it is impossible to obtain an estimate of the modulus of the logarithmic derivatives uniformly in $\arg z$ for such functions.

A. Z. Mokhon'ko proved that

$$\left|\frac{f'(z)}{f(z)}\right| < \frac{K|z|^{2(\rho+1+\varepsilon)}}{\sin^2\varphi},\ z = re^{i\varphi} \in \mathbb{C}\setminus E, \tag{7}$$

where E is the set of disks with finite sum of radii, K is some constant.

For meromorphic functions $f(z), z \in \mathbb{C}$ one can deduce from the inequality (1) the estimate for $|f^{(n)}(z)/f(z)|$, using the equality

$$\left|\frac{f^{(n)}}{f}\right| = \left|\frac{f^{(n)}}{f^{(n-1)}}\right|\left|\frac{f^{(n-1)}}{f^{(n-2)}}\right|\dots\left|\frac{f'}{f}\right|.$$

This method relies implicitly on a theorem stating that the growth category of f is not less than the growth category of f' [12, p. 131, Theorem 2.3]. In its turn, in the proof of this theorem the logarithmic derivative lemma is used, which has no counterpart for the half-plane (see the previous remark).

Thus, we skip "quick" proof and start with an estimate for $|d^n \log f(z)/dz^n|$, which is of independent interest.

We apply the more powerful method of light points, which allows us to obtain better results. Using Theorem 2, we deduce an estimate of the logarithmic derivative via growth characteristics of the function f outside an exceptional set. The estimates and the size of exceptional set are related via a function-parameter ψ. It is impossible to avoid exceptional sets, because the logarithmic derivative is unbounded in any neighborhood of a zero or a pole of the function.

We denote $D(z,\sigma)=\{\zeta\in\mathbb{C}:|\zeta-z|<\sigma\}$.

Theorem 1. *Let $f(z)$, $z\in\mathbb{C}_+(r_0)$ be a meromorphic function. Let $(r_\nu)_{\nu=1}^{\infty}$ be an increasing to $+\infty$ sequence and $\psi\colon(0,+\infty)\to(0,+\infty)$ be such that $\psi(r_{\nu+1})=O(r_{\nu-1})$ $(\nu\to+\infty)$. Then for $z=re^{i\varphi}$, $0<\varphi<\pi$,*

$$\Big|\frac{d^n\log f(z)}{dz^n}\Big|<\frac{Kr_\nu^2S(r_{\nu+1},f)}{(r_{\nu-1}-r_{\nu-2})^{n+1}\sin^{n+1}\varphi}+\frac{W(z)}{\sin^n\varphi},\quad r_{\nu-1}<r\le r_\nu,$$

where

$$|W(z)|\le\begin{cases}K\dfrac{c(r_{\nu+1},0,\infty)}{\psi(r_{\nu+1})}\log\dfrac{c(r_{\nu+1},0,\infty)r_{\nu+1}}{\psi(r_{\nu+1})\sin\varphi}, & n=1,\\ K\Big(\frac{c(r_{\nu+1},0,\infty)}{\psi(r_{\nu+1})\sin\varphi}\Big)^n, & n>1,\end{cases},\quad z\in\mathbb{C}_+(r_0)\setminus E, \tag{8}$$

for some set $E\subset\bigcup_j D(z_j,\sigma_j)$, a locally finite covering $\{D(z_j,\sigma_j)\}_{j\ge1}$, and a constant K depending on n and r_0. Moreover, there exists $\nu_0\in\mathbb{N}$ such that

$$\sum_{|z_j|\le r_\nu}\sigma_j<12\sum_{k=\nu_0}^{\nu+1}\psi(r_k),\quad \nu\to+\infty. \tag{9}$$

Choosing $\psi(r)=\varepsilon r^\tau$, where $\tau\in(0,1]$, $1\ge\varepsilon>0$, we obtain the following corollary.

Corollary 1. *For any $\varepsilon\in(0,1]$, $\gamma>1$, $\tau\in(0,1]$, $z=re^{i\varphi}$, $0<\varphi<\pi$,*

$$\Big|\frac{d^n\log f(z)}{dz^n}\Big|<\frac{K\gamma^{n+2}S(\gamma r,f)}{(\gamma^{\frac12}-1)^{n+1}r^{n-1}\sin^{n+1}\varphi}+\frac{W(z)}{\sin^n\varphi},$$

and

$$|W(z)|\le\begin{cases}\dfrac{K}{\varepsilon}\dfrac{c(\gamma r,0,\infty)}{r^\tau}\log\dfrac{c(\gamma r,0,\infty)r^{1-\tau}}{\varepsilon\sin\varphi}, & n=1,\\ \frac{K}{\varepsilon}\Big(\frac{c(\gamma r,0,\infty)}{r^\tau\sin\varphi}\Big)^n, & n>1,\end{cases},\quad z\in\mathbb{C}_+(r_0)\setminus E_\tau, \tag{10}$$

where $E_\tau\subset\bigcup_j D(z_j,\sigma_j)$, the covering $\{D(z_j,\sigma_j)\}_{j\ge1}$ is locally finite, the constant K depends on n, r_0 and γ,

$$\sum_{|z_j|\le R}\sigma_j<\varepsilon K_0(\gamma)R^\tau,\quad r\to+\infty, \tag{11}$$

with $K_0(\gamma)=\dfrac{12\gamma^{\frac{\tau}{2}}}{1-\gamma^{-\frac{\tau}{2}}}$. In particular, if $S(r,f)$ has order ρ, then

$$\Big|\frac{d^n\log f(z)}{dz^n}\Big|\le\Big(\frac{|z|^{\rho+1-\tau+\varepsilon}}{\sin^2\varphi}\Big)^n,\quad z\notin E_\tau.$$

Corollary 2. *If we write $\psi(r) = (\log r)^{-1-\delta}$, $\delta > 0$, then $W(z)$ allows the following estimate for $z = re^{i\varphi} \in \mathbb{C}_+(r_0) \setminus E_0$, $0 < \varphi < \pi$ and some positive constant $K(\delta) > 0$:*

$$|W(z)| \leq \begin{cases} Kc(\gamma r, 0, \infty)(\log r)^{1+\delta} \log \dfrac{c(\gamma r, 0, \infty) r \log^{1+\delta} r}{\sin \varphi}, & n = 1, \\ K\Big(\dfrac{c(\gamma r, 0, \infty)(\log r)^{1+\delta}}{\sin \varphi}\Big)^n, & n > 1, \end{cases} \tag{12}$$

where $E_0 \subset \bigcup_j D(z_j, \sigma_j)$, the covering $\{D(z_j, \sigma_j)\}_{j\geq 1}$ is locally finite, and

$$\sum_j \sigma_j < K(\delta). \tag{13}$$

In particular, if $S(r, f)$ is of order ρ, then

$$\Big|\frac{d^n \log f(z)}{dz^n}\Big| \leq \Big(\frac{|z|^{\rho+1+\varepsilon}}{\sin^2 \varphi}\Big)^n, \quad z \notin E_0, \tag{14}$$

$$\Big|\frac{f^{(n)}(z)}{f(z)}\Big| \leq \Big(\frac{|z|^{\rho+1+\varepsilon}}{\sin^2 \varphi}\Big)^n, \quad z \notin E_0, \tag{15}$$

where E_0 satisfies (13).

Without loss of generality we assume that $f(r_0 e^{i\theta}) \neq 0, \infty;\ 0 \leqslant \theta \leqslant \pi$. Otherwise, we can enlarge a bit r_0 such that this assumption holds.

The proof of Theorem 1 relies on the following theorem.

Theorem 2. *Let $f(z)$, $z \in \mathbb{C}_+(r_0)$ be a meromorphic function. If $z = re^{i\varphi}$, $r_0 < |z| < s$, $\operatorname{Im} z > 0$, then $\forall n \in \mathbb{N}$*

$$\Big|\frac{d^n \log f(z)}{dz^n}\Big| < \frac{Ks^2 S(s, f)}{(s-r)^{n+1} \sin^{n+1} \varphi} \Big(\frac{s}{r}\Big)^n +$$

$$+\frac{K}{\sin^n \varphi} \sum_{r_0 < |c_q| < s} \left(\frac{\sin \theta_q}{(s-r)^n} + \frac{\sin \theta_q}{|z - c_q|^n}\right) + K,\ K = \text{const} > 0, \tag{16}$$

where $S(r, f)$ is the Nevanlinna characteristic of $f(z)$, $z \in \mathbb{C}_+(r_0)$ (3), $c_q = |c_q| \exp(i\theta_q) \in \mathcal{Z}_f \cup \mathcal{P}_f$, $\mathcal{P}_f$ and $\mathcal{Z}_f$ stand for the set of poles and the zero set of f, respectively.

Remark 1. If $f(z)$, $z \in \mathbb{C}_+(r_0)$ has finite order ρ, then for arbitrary $\varepsilon > 0$ the previous theorem implies

$$\Big|\frac{d^n \log f(z)}{dz^n}\Big| < \frac{K|z|^{(n+1)(\rho+1+\varepsilon)}}{\sin^{2n} \varphi}, \ z \notin E, \tag{17}$$

where $E = \bigcup_q D(c_q, |c_q|^{-\rho-1-\varepsilon} \sin \theta_q)$, is a set of disks with finite sum of radii centered at zeros and poles of f, $n \in \mathbb{N}$.

Combining (17) and Lemma 1, it is not hard to get

$$\Big|\frac{f^{(n)}(z)}{f(z)}\Big| < \frac{K|z|^{2n(\rho+1+\varepsilon)}}{\sin^{2n} \varphi}, \ z \notin E. \tag{18}$$

Remark 2. If $n = 1$ both estimates (17), (18) coincide. If $n = 2, 3, 4, ...$, then they are different.

Remark 3. For a meromorphic function $f(z)$ in the angular domain $\{z : \alpha \leqslant \arg z \leqslant \beta, |z| > r_1\}$, relationships similar to (16)–(18) can be obtained by application Theorem 2 to the function $f_1(z) = f(z^{1/k}e^{i\alpha})$, $k = \pi/(\beta - \alpha)$, meromorphic in the closed domain $\{z : \operatorname{Im} z \geqslant 0, |z| \geqslant r_0\}$ (see [12, p. 41]). Note that estimates (16), (18) depend on $\varphi = \arg z$.

One can apply estimate (18) in complex differential equations as Valiron's inequality ([19]) was applied. Estimate (18) can be also used for investigation of asymptotic properties of solutions in a neighborhood of a logarithmic singularity or in an angular domain.

Remark 4. Unlike estimates (17) and (18) interplay between the system of disks $D(z_j, \sigma_j)$ covering an exceptional set from Theorem 1 on one hand and the set of zeros and poles $\mathcal{Z}_f \cup \mathcal{P}_f$ of f, on the other hand, is more complicated. In particular, z_j not necessary belongs to $\mathcal{Z}_f \cup \mathcal{P}_f$. However, we can always assume that every connected component of the open set $U = \bigcup_j D(z_j, \sigma_j)$ contains a point from $\mathcal{Z}_f \cup \mathcal{P}_f$ (see [3, p.123]).

1. Preliminaries. We need some lemmas.

Lemma 1. *Let $f(z)$, $z \in D$ be a meromorphic function in a domain D. Then we have for $n = 2, 3, \dots$*

$$\frac{f^{(n)}(z)}{f(z)} = \left(\frac{f'(z)}{f(z)}\right)^n + \sum_{\sum q i_q = n} B_{i_1 \dots i_{n-1}} \prod_{q=1}^{n-1} \left(\frac{d^q \log f(z)}{dz^q}\right)^{i_q} + \frac{d^n \log f(z)}{dz^n}, \tag{19}$$

where we sum up over all $i_1, \dots, i_{n-1}$ such that $0 \leqslant i_1, \dots, i_{n-1} < n$, $\sum q i_q = 1i_1 + 2i_2 + \dots + (n-1)i_{n-1} = n$; $B_{i_1 \dots i_{n-1}}$ are nonnegative.

Proof of Lemma 1. Branches of the multivalued analytic function $\operatorname{Log} f(z)$, $z \in D$, are holomorphic in some small neighborhood of z, $z \neq c_q \in \mathcal{Z}_f \cup \mathcal{P}_f$. If a disk $\{\eta : |\eta| < \delta\}$ is such that $\{\eta : |\eta| < \delta\} \cap (\mathcal{Z}_f \cup \mathcal{P}_f) = \varnothing$, then any branch $\log f(z)$ of $\operatorname{Log} f(z)$ in this disk can be represented as a convergent series

$$\log f(z + \eta) - \log f(z) = \sum_{j=1}^{\infty} \frac{1}{j!} \frac{d^j \log f(z)}{dz^j} \eta^j. \tag{20}$$

Thus

$$f(z + \eta) = f(z) \exp\Big\{\sum_{j=1}^{\infty} \frac{1}{j!} \frac{d^j \log f(z)}{dz^j} \eta^j\Big\}, \quad |\eta| < \delta. \tag{21}$$

Consider the series

$$f(z + \eta) = f(z) + \sum_{j=1}^{\infty} \frac{1}{j!} f^{(j)}(z) \eta^j. \tag{22}$$

It follows from (21) that

$$f(z + \eta) = f(z) \exp\left\{\sum_{j=1}^{\infty} \frac{\eta^j}{j!} \frac{d^j \log f(z)}{dz^j}\right\} =$$

$$= f(z)\sum_{k=0}^{\infty}\frac{1}{k!}\left\{\sum_{j=1}^{\infty}\frac{\eta^j}{j!}\frac{d^j\log f(z)}{dz^j}\right\}^k = f(z)\sum_{n=0}^{\infty}A_n\eta^n. \tag{23}$$

Direct computations show that $A_1 = f'(z)/f(z)$. For $n = 2, 3, 4, \dots$,

$$A_n = \frac{1}{n!}\left(\frac{f'(z)}{f(z)}\right)^n + \sum_{i_1,\dots,i_{n-1}} B_{i_1,\dots,i_{n-1}}\prod_{q=1}^{n-1}\left(\frac{d^q\log f(z)}{dz^q}\right)^{i_q} + \frac{1}{n!}\frac{d^n\log f(z)}{dz^n}, \tag{24}$$

where we sum up over all integers $i_1, i_2, \dots, i_{n-1}$ satisfying $0 \leqslant i_1, i_2, \dots,\ i_{n-1} < n,\ 1i_1 + 2i_2 + \dots + (n-1)i_{n-1} = n$; the constants $B_{i_1\dots i_{n-1}}$ are nonnegative. Relations (22), (23) imply $A_n f(z) = \frac{1}{n!}f^{(n)}(z)$, consequently, (24) (after the reassignment of the coefficients) yields (19). □

Suppose that $f(z)$ $z \in \mathbb{C}_+(r_0)$ is meromorphic, $c_q = |c_q|\exp(i\theta_q) \in \mathcal{Z}_f \cup \mathcal{P}_f$.

Lemma 2. *Let $R > s > r_0$. Then*

$$c(s,0,\infty) \leqslant \frac{R^2 sC(R,0,\infty)}{2(R-s)(R+s)}. \tag{25}$$

Proof of Lemma 2. It follows from the definition of $C(r,f)$ that

$$C(R,0,\infty) = C(R,f) + C(R,f^{-1}) \geqslant 2\int_s^R c(t,0,\infty)\left(\frac{1}{t^2}+\frac{1}{R^2}\right)dt \geqslant$$

$$\geqslant 2c(s,0,\infty)\int_s^R\left(\frac{1}{t^2}+\frac{1}{R^2}\right)dt = \frac{2(R-s)(R+s)}{R^2 s}c(s,0,\infty).$$

□

Let

$$F(z,\zeta) = \log[(s^2 - z\overline{\zeta})(z-\overline{\zeta})(z-\zeta)^{-1}(s^2 - z\zeta)^{-1}], \tag{26}$$

$z, \zeta \in U = \{z : r_0 \leqslant |z| \leqslant s, \operatorname{Im} z \geqslant 0\}$, $z \neq \zeta$, be a branch of the multivalued function, which will be specified in each case.

The derivation operator by internal normal to the boundary of U applied to $\operatorname{Re} F(z,\zeta)$ and $f(\zeta)$ with respect to the variable ζ is denoted by $\partial/\partial\eta$.

The following Nevanlinna formula was established in [18].

Lemma 3. *Let $f(z) \not\equiv 0$ be meromorphic in $U = \{z : r_0 \leqslant |z| \leqslant s,\ \operatorname{Im} z \geqslant 0\}$, and $f(z) \neq 0, \infty$ for $|z| = r_0$. Then, for any simply connected open set $V \subset U$ satisfying $V \cap (\mathcal{Z}_f \cup \mathcal{P}_f) = \varnothing$ and any branch $\log f$ of $\operatorname{Log} f$ in V there exists a real constant C such that for all $z \in V$ we have*

$$\log f(z) = \frac{i}{2\pi}\int\limits_{[-s,-r_0]\cup[r_0,s]}\log|f(t)|\left[\frac{t+z}{t-z} - \frac{s^2+tz}{s^2-tz}\right]\frac{dt}{t} +$$

$$+\frac{1}{2\pi}\int_0^{\pi}\log|f(\zeta)|\left[\frac{\zeta+z}{\zeta-z} - \frac{\overline{\zeta}+z}{\overline{\zeta}-z}\right]_{\zeta=se^{i\theta}}d\theta - \tag{27}$$

$$-\sum_{r_0<|a_m|<s}F(z,a_m) + \sum_{r_0<|b_l|<s}F(z,b_l) + Q(z,s),$$

$$Q(z,s)=\frac{r_0}{2\pi}\int\limits_0^{\pi}\left[\log|f(\zeta)|\frac{\partial F(z,\zeta)}{\partial\eta}-F(z,\zeta)\frac{\partial\log|f(\zeta)|}{\partial\eta}\right]_{\zeta=r_0e^{i\theta}}d\theta+iC, \tag{28}$$

where a_m are zeros, b_l poles of $f(z)$, counted according their multiplicities.

Remark 5. *In* (27) *$F(z,a_m)$ and $F(z,b_l)$ denote some branches in V. In the integral* (28) *we first choose an arbitrary branch $F(z,r_0)$, $z\in V$, then choose a branch on the semi-circle $|\zeta|=r_0$, $0\le\arg\zeta\le\pi$ which coincides with the previous one at $\zeta=r_0$. It is possible, because $f(\zeta)\ne 0,\infty$ when $|\zeta|=r_0$.*

2. Proofs of the theorems.

Proof of Theorem 2. We differentiate (27) n times by z:

$$\frac{d^n}{dz^n}\log f(z)=\frac{i}{2\pi}\int\limits_{[-s,-r_0]\cup[r_0,s]}\log|f(t)|\left[\frac{t+z}{t-z}-\frac{s^2+tz}{s^2-tz}\right]_z^{(n)}\frac{dt}{t}+$$

$$+\frac{1}{2\pi}\int\limits_0^{\pi}\log|f(\zeta)|\left(\left[\frac{\zeta+z}{\zeta-z}-\frac{\overline{\zeta}+z}{\overline{\zeta}-z}\right]_{\zeta=se^{i\theta}}\right)_z^{(n)}d\theta-$$

$$-\sum_{r_0<|a_m|<s}F_z^{(n)}(z,a_m)+\sum_{r_0<|b_l|<s}F_z^{(n)}(z,b_l)+Q_z^{(n)}(z,s). \tag{29}$$

The following equalities are valid:

$$\left[\frac{t+z}{t-z}-\frac{s^2+tz}{s^2-tz}\right]_z^{(n)}=\frac{2tn!}{(t-z)^{n+1}}-\frac{2s^2t^nn!}{(s^2-zt)^{n+1}}, \tag{30}$$

$$\left(\frac{\zeta+z}{\zeta-z}-\frac{\overline{\zeta}+z}{\overline{\zeta}-z}\right)_z^{(n)}=\frac{2\zeta n!}{(\zeta-z)^{n+1}}-\frac{2\overline{\zeta}n!}{(\overline{\zeta}-z)^{n+1}}, \tag{31}$$

$$(F(z,\zeta))_z^{(n)}=-\frac{(\overline{\zeta})^n(n-1)!}{(s^2-z\overline{\zeta})^n}-\frac{(n-1)!}{(\overline{\zeta}-z)^n}+\frac{(n-1)!}{(\zeta-z)^n}+\frac{\zeta^n(n-1)!}{(s^2-z\zeta)^n}. \tag{32}$$

On the arc $\{\zeta:\zeta=r_0\exp(i\theta),0<\theta<\pi\}$ the derivative by the internal normal at the point $\zeta=\rho e^{i\theta}$ has the form $\partial F(z,\zeta)/\partial\eta=\partial F(z,\rho\exp(i\theta))/\partial\rho$. Therefore,

$$\left.\frac{\partial F}{\partial\eta}\right|_{\zeta=r_0e^{i\theta}}=\frac{-z}{s^2e^{i\theta}-zr_0}+\frac{1}{ze^{-i\theta}-r_0}+\frac{z}{s^2e^{-i\theta}-zr_0}-\frac{1}{ze^{i\theta}-r_0}. \tag{33}$$

Differentiating (33) n times by z we arrive to

$$\frac{1}{n!}\left(\frac{\partial F(z,r_0e^{i\theta})}{\partial\eta}\right)_z^{(n)}=-\frac{s^2e^{i\theta}r_0^{n-1}}{(s^2e^{i\theta}-zr_0)^{n+1}}-\frac{e^{-in\theta}}{(r_0-ze^{-i\theta})^{n+1}}+$$

$$+\frac{s^2e^{-i\theta}r_0^{n-1}}{(s^2e^{-i\theta}-zr_0)^{n+1}}+\frac{e^{in\theta}}{(r_0-ze^{i\theta})^{n+1}}. \tag{34}$$

Let

$$r_0+1<|z|<s,\quad s>\max(2r_0,r_0+1). \tag{35}$$

Then $|s^2e^{i\theta}-zr_0|$, $|s^2e^{-i\theta}-zr_0|>s^2-sr_0$; $|r_0-ze^{-i\theta}|>1$, $|r_0-ze^{i\theta}|>1$. These inequalities and (34) yield

$$\left|\left(\frac{\partial F(z,r_0e^{i\theta})}{\partial\eta}\right)_z^{(n)}\right|<4n!. \tag{36}$$

Suppose that (35) holds and $|\zeta|=r_0$. Then $|z-\overline{\zeta}|>1$, $|z-\zeta|>1$; $|s^2-z\overline{\zeta}|>s^2-sr_0=s(s-r_0)>sr_0$, $|s^2-z\zeta|>s^2-sr_0>sr_0$, and (32) allows the estimate

$$\left|(F(z,\zeta))_z^{(n)}\right|_{|\zeta|=r_0}<4(n-1)! \tag{37}$$

Since $f(r_0e^{i\theta})\neq 0,\infty$, we have $|\log|f(r_0e^{i\theta})||<K$, $|\partial\log|f(\zeta)|/\partial\eta|_{\zeta=r_0e^{i\theta}}<K$, $0\le\theta\le\pi$, $K=\mathrm{const}$. Thus, (28), (36), (37), imply

$$|(Q(z,s))_z^{(n)}|<4r_0Kn!,\qquad K=\mathrm{const}, \tag{38}$$

where K does not depend on z and s.

Using the binomial formula and elementary calculation, we rewrite (31) in the form

$$\frac{1}{2n!}\left(\frac{\zeta+z}{\zeta-z}-\frac{\overline{\zeta}+z}{\overline{\zeta}-z}\right)_z^{(n)}=\frac{\sum_{j=0}^{n+1}C_{n+1}^j(-z)^{n+1-j}(\zeta\overline{\zeta}^j-\overline{\zeta}\zeta^j)}{(\zeta-z)^{n+1}(\overline{\zeta}-z)^{n+1}}. \tag{39}$$

Since $x^n-y^n=(x-y)(x^{n-1}+x^{n-2}y+\cdots+xy^{n-2}+y^{n-1})$, the numerator in the right-hand side of (39) can be represented as follows

$$(-z)^{n+1}(\zeta-\overline{\zeta})+\sum_{j=2}^{n+1}C_{n+1}^j(-z)^{n+1-j}|\zeta|^2(\overline{\zeta}^{j-1}-\zeta^{j-1})=$$

$$=(\zeta-\overline{\zeta})\left((-z)^{n+1}-\sum_{j=2}^{n+1}C_{n+1}^j(-z)^{n+1-j}|\zeta|^2\sum_{p=0}^{j-2}\overline{\zeta}^p\zeta^{j-2-p}\right). \tag{40}$$

If $\zeta=se^{i\theta}$, then $\zeta-\overline{\zeta}=2i|\zeta|\sin\theta$. Therefore, combining (40) and the formulas

$$\sum_{j=0}^{n}jC_n^j=n2^{n-1},\qquad \sum_{j=0}^{n}C_n^j=2^n \tag{41}$$

we estimate the numerator in the right-hand side of (39) ($|z|<|\zeta|=s$),

$$\left|\sum_{j=0}^{n+1}C_{n+1}^j(-z)^{n+1-j}|\zeta|^2(\overline{\zeta}^{j-1}-\zeta^{j-1})\right|<$$

$$<2\sin\theta s^{n+2}\Big(1+\sum_{j=2}^{n+1}C_{n+1}^j(j-1)\Big)<2\sin\theta s^{n+2}(n2^n+2).$$

Hence, taking into account (39), and the inequalities $|\overline{\zeta}-z|>s\sin\varphi$, $|\zeta-z|>s-r$, we have

$$\left|\left(\frac{\zeta+z}{\zeta-z}-\frac{\overline{\zeta}+z}{\overline{\zeta}-z}\right)_z^{(n)}\right|_{\zeta=se^{i\theta}}<\frac{Ks\sin\theta}{(s-r)^{n+1}\sin^{n+1}\varphi}, \tag{42}$$

$K = 4n!(n2^n + 2)$ does not depend on s. The expression (30) is estimated similarly:

$$\frac{1}{2n!}\left(\frac{t+z}{t-z}-\frac{s^2+tz}{s^2-tz}\right)^{(n)}_z = \frac{t(s^2-zt)^{n+1}-s^2t^n(t-z)^{n+1}}{(t-z)^{n+1}(s^2-zt)^{n+1}}. \tag{43}$$

We write the numerator on the right-hand side of (43) in the form

$$t\sum_{j=0}^{n+1} C^j_{n+1}(-zt)^{n+1-j}s^{2j} - s^2t^n\sum_{j=0}^{n+1} C^j_{n+1}(-z)^{n+1-j}t^j =$$

$$= \sum_{j=0}^{n+1} C^j_{n+1}(-z)^{n+1-j}(t^{n+2-j}s^{2j} - s^2t^{n+j}) =$$

$$= ts^2\left(\sum_{j=2}^{n+1} C^j_{n+1}(-zt)^{n+1-j}\left((s^2)^{j-1}-(t^2)^{j-1}\right)\right) - (-z)^{n+1}t^n(s^2-t^2) =$$

$$= (s^2-t^2)\left(ts^2\sum_{j=2}^{n+1} C^j_{n+1}(-zt)^{n+1-j}\sum_{k=0}^{j-2}(s^2)^{j-2-k}(t^2)^k - (-z)^{n+1}t^n\right). \tag{44}$$

Since $|z| \leqslant s, |t| \leqslant s$, using (44), (41), we deduce

$$|t(s^2-zt)^{n+1}-s^2t^n(t-z)^{n+1}| < (s^2-t^2)|t|s^{2n}(n2^n+2). \tag{45}$$

For $z = re^{i\varphi}, t \in \mathbb{R}, |t| \leqslant s$, we obtain

$$|s^2-zt| \geqslant s(s-r), \quad |z-t| \geqslant r\sin\varphi, \quad |z-t| \geqslant |t|\sin\varphi. \tag{46}$$

Thus, (43), (45), (46) imply

$$\left|\left(\frac{t+z}{t-z}-\frac{s^2+tz}{s^2-tz}\right)^{(n)}_z\right| \leqslant \frac{|t|s^{n+2}2n!(n2^n+2)}{r^n(s-r)^{n+1}\sin^{n+1}\varphi}\left(\frac{1}{t^2}-\frac{1}{s^2}\right). \tag{47}$$

We write (32) in the following form

$$\frac{(F(z,\zeta))^{(n)}_z}{(n-1)!} = \left(\frac{\zeta^n}{(s^2-z\zeta)^n}-\frac{\overline{\zeta}^n}{(s^2-z\overline{\zeta})^n}\right)+\left(\frac{1}{(\zeta-z)^n}-\frac{1}{(\overline{\zeta}-z)^n}\right). \tag{48}$$

We reduce the expressions on the right-hand side of (48) to common denominator. Since $\zeta = |\zeta|\exp(i\theta)$; $z = r\exp(i\varphi)$; $|\zeta|, |z| < s$, we get

$$|s^2-z\zeta|,\ |s^2-z\overline{\zeta}| > s(s-r),\ |\overline{\zeta}-z| > r\sin\varphi. \tag{49}$$

We then show that at least one of the inequalities

$$|s^2-z\zeta| > s^2\sin\varphi, \quad |s^2-z\overline{\zeta}| > s^2\sin\varphi$$

holds. In fact, since $0<\varphi<\pi,\ 0<\theta<\pi$, the points $z\zeta = r|\zeta|e^{i(\varphi+\theta)}$, $z\overline{\zeta} = r|\zeta|e^{i(\varphi-\theta)}$ lie in different half-planes with respect to the line $\Lambda = \{z : z = te^{i\varphi},\ \varphi = \text{const},\ -\infty < t < +\infty\}$. Without loss of generality, we may assume that the points s^2 and $z\overline{\zeta}$ lie from both sides of the line Λ. Then $|s^2 - z\overline{\zeta}|$ is greater than the distance from s^2 to Λ. Therefore, $|s^2 - z\overline{\zeta}| > s^2 \sin\varphi$. Combining this with (49) and with the equalities $x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$, $\zeta - \overline{\zeta} = 2i|\zeta|\sin\theta$, we obtain the following estimate for (48)

$$\left|(F(z,\zeta))_z^{(n)}\right| < n!2^n \frac{\sin\theta}{\sin^n\varphi}\left(\frac{1}{(s-r)^n} + \frac{1}{|\zeta - z|^n}\right). \tag{50}$$

It follows from (29), (38), (42), (47), (50) that

$$\frac{1}{K}\left|\frac{d^n \log f(z)}{dz^n}\right| < \frac{s^{n+2}}{\pi r^n (s-r)^{n+1}\sin^{n+1}\varphi} \int\limits_{[-s,-r_0]\cup[r_0,s]} |\log|f(t)||\left(\frac{1}{t^2} - \frac{1}{s^2}\right)dt +$$

$$+ \frac{s^2}{(s-r)^{n+1}\sin^{n+1}\varphi}\frac{1}{\pi s}\int\limits_0^\pi |\log|f(se^{i\theta})||\sin\theta d\theta +$$

$$+ \frac{1}{\sin^n\varphi}\sum_{r_0<|c_q|<s}\left(\frac{\sin\theta_q}{(s-r)^n} + \frac{\sin\theta_q}{|z-c_q|^n}\right) + 1,\ K = \text{const.} \tag{51}$$

By the definition of the Nevanlinna characteristic (3), (5), and (51) we obtain (16). $\square$

Proof of Theorem 1. We denote

$$W(z) := \frac{K}{\sin^n\varphi}\sum_{r_0<|c_q|<s}\left(\frac{\sin\theta_q}{(s-r)^n} + \frac{\sin\theta_q}{|z-c_q|^n}\right) + K,$$

where K is the constant from (16). Then, according to Theorem 2 we have

$$\left|\frac{d^n \log f(z)}{dz^n}\right| < \frac{Ks^2 S(s,f)}{(s-r)^{n+1}\sin^{n+1}\varphi}\left(\frac{s}{r}\right)^n + W(z), \tag{52}$$

$z = re^{i\varphi} \in \mathbb{C}_+(r_0)$, $r < s$, $0 < \varphi < \pi$.

We define

$$G_\nu^* = \{\zeta \in \mathbb{C}_+(r_0) : r_{\nu-1} \le |\zeta| \le r_\nu, \operatorname{Im}\zeta \ge 0\},$$

$$G_\nu = \{\zeta \in \mathbb{C}_+(r_0) : r_{\nu-2} < |\zeta| \le r_{\nu+1}, \operatorname{Im}\zeta \ge 0\}, \quad \nu \ge \nu_0,$$

where ν_0 is the least natural satisfying the inequality $2^{\nu_0} \ge r_0$. For a fixed z we denote also $G_\nu^+(z) = \{\zeta \in G_\nu : \varphi_1/2 \le \arg\zeta \le \pi - \varphi_1/2\}$, where $\varphi_1 = \min\{\varphi, \pi - \varphi\}$, $z = re^{i\varphi}$, $G_\nu^-(z) = G_\nu \setminus G_\nu^+(z)$.

We then define a discrete measure λ_ν, $\nu \ge \nu_0$ on $\mathbb{C}$ in the following way

$$\lambda_\nu(F) = \sum_{c_q \in F \cap G_\nu} \sin\theta_q,$$

where $F \subset \mathbb{C}$. A point $z \in G_\nu^*$ is called *light*, if for an arbitrary $\sigma > 0$

$$\lambda_z(\sigma) \stackrel{def}{=} \lambda_\nu(\overline{D}(z,\sigma)) = \sum_{\substack{|c_k - z| \le \sigma \\ c_k \in G_\nu}} \sin\theta_k < b_\nu\sigma, \tag{53}$$

where $b_\nu = \frac{c(r_{\nu+1},0,\infty)}{\psi(r_{\nu+1})}$. We may suppose that $\lambda_\nu(\overline{G}_\nu) > 0$, otherwise all points of G_ν^* are light. If z is not light, we say that it is *heavy*, i.e. for some $\sigma_z > 0$ one has $\lambda_\nu(\bar{D}(z,\sigma_z)) \ge b_\nu\sigma_z$.

Next, we show that the set of light points in G_ν^* is open in in G_ν^*. For fixed $\nu \in \mathbb{N}$, $z \in G_\nu^*$ consider the function $\lambda_z(\sigma)$. Since λ_ν is a discrete measure with $\operatorname{supp}\lambda_\nu \subset \overline{G}_\nu$ and $z \in G_\nu^*$, $\lambda_z(\sigma)$ is a nondecreasing step function with finitely many jumps at points $0 < \sigma_1 < \sigma_2 < \cdots < \sigma_m \le r_\nu + r_{\nu+1}$, where the number of jumps m depends only on z, ν and f. Moreover, λ_z is continuous from the right, and $\lambda_z(\sigma) = 0$, $0 \le \sigma < \sigma_1$ provided that z is light.

Therefore, by (53) and the definition of σ_j, $1 \le j \le m$,

$$\sup_{\sigma>0} \frac{\lambda_z(\sigma)}{\sigma} = \max_{1\le j\le m} \frac{\lambda_z(\sigma_j)}{\sigma_j} = D_z < b_\nu.$$

We choose $\delta \in (0, (1 - \frac{D_z}{b_\nu}))\sigma_1$, so that $\frac{\sigma_1}{\sigma_1-\delta} < \frac{b_\nu}{D_z}$. Then for any $w \in \overline{D}(z,\delta)$ and $\sigma > 0$ we have

$$\begin{gathered}\frac{\lambda_\nu(\bar{D}(w,\sigma))}{\sigma} \le \frac{\lambda_\nu(\bar{D}(z,\sigma+\delta))}{\sigma} = \frac{\lambda_z(\sigma+\delta)}{\sigma} \le \\ \le \sup_{\sigma>0} \frac{\lambda_z(\sigma+\delta)}{\sigma+\delta}\frac{\sigma+\delta}{\sigma} \le D_z\frac{\sigma_1}{\sigma_1-\delta} < b_\nu.\end{gathered}$$

So, w is a light point, and consequently the set of light points from G_ν^* is open in G_ν^*.

We cover all heavy points z by open disks of the radius $2\sigma_z$ centered at z. Then, by Ahlfors-Landkof lemma [16, Chap.III, Lemma 3.2, p.246], there exists an at most countable subcovering of multiplicity not greater than 6 by disks $D(z_{k,\nu}, 2\sigma_{z_{k,\nu}})$. Taking into account that the set of heavy points is closed in G_ν^*, as the complement to an open set, we conclude that the set of heavy points is compact. Therefore, there exists a finite subcovering $\{D(z_{k,\nu}, 2\sigma_{z_{k,\nu}})\}_{k=1}^{N_\nu}$. Further, by the definition of a light point we deduce

$$\begin{gathered}\sum_{k=1}^{N_\nu}\sigma_{z_{k,\nu}} \le \frac{1}{b_\nu}\sum_{k=1}^{N_\nu}\lambda_\nu(\bar{D}(z_{k,\nu},\sigma_{z_{k,\nu}})) \le \frac{6}{b_\nu}\lambda_\nu\Big(\bigcup_{k=1}^{N_\nu} D(z_{k,\nu}, 2\sigma_{z_{k,\nu}})\Big) \le \\ \le \frac{6}{b_\nu}\lambda_\nu(G_\nu) \le \frac{6}{b_\nu}c(r_{\nu+1},0,\infty) = 6\psi(r_{\nu+1}).\end{gathered}$$

Let us estimate the exceptional set of heavy points. For $R \in (r_{m-1}, r_m]$, $m \in \mathbb{N}$ we have

$$\sum_{|z_{k,\nu}|\le R}\sigma_{k,\nu} \le \sum_{\nu=\nu_0}^{m}\sum_{k=1}^{N_\nu}\sigma_{z_{k,\nu}} \le \sum_{\nu=\nu_0}^{m} 6\psi(r_{\nu+1}). \tag{54}$$

Let z be light. We write the sum

$$\sum_{c_q\in G_\nu}\frac{\sin\theta_q}{|z-c_q|^n} = \Bigg(\sum_{c_q\in G_\nu^+(z)} + \sum_{c_q\in G_\nu^-(z)}\Bigg)\frac{\sin\theta_q}{|z-c_q|^n} \equiv \sum_1 + \sum_2. \tag{55}$$

It follows from the definition of a light point from G_ν^* that $\sin\theta_q \le b_\nu|z-c_q|$. Thus, in the first sum in (55) we have

$$2r_{\nu+1} \ge |z-c_q| \ge \frac{\sin\theta_q}{b_\nu} \ge \frac{\sin(\varphi_1/2)}{b_\nu}.$$

Since $\lambda_z(t) < b_\nu t$, by properties of the Stieltjes integral

$$\sum_{c_q \in G_\nu^+(z)} \frac{\sin\theta_q}{|z-c_q|^n} \le \int\limits_{G_\nu^+(z)} \frac{d\lambda_\nu(\tau)}{|\tau - z|^n} \le \int\limits_{\frac{\sin\frac{\varphi_1}{2}}{b_\nu}}^{2r_{\nu+1}} \frac{d\lambda_z(t)}{t^n} =$$

$$= \frac{\lambda_z(t)}{t^n}\Bigg|_{\frac{\sin\frac{\varphi_1}{2}}{b_\nu}}^{2r_{\nu+1}} + \int\limits_{\frac{\sin\frac{\varphi_1}{2}}{b_\nu}}^{2r_{\nu+1}} \frac{n\lambda_z(t)}{t^{n+1}}dt < \frac{b_\nu}{(2r_{\nu+1})^{n-1}} + \int\limits_{\frac{\sin\frac{\varphi_1}{2}}{b_\nu}}^{2r_{\nu+1}} \frac{nb_\nu}{t^n}dt.$$

If $n > 1$ the last estimate yields

$$\sum_{c_q \in G_\nu^+(z)} \frac{\sin\theta_q}{|z-c_q|^n} \le \frac{b_\nu}{(2r_{\nu+1})^{n-1}} - \frac{n}{n-1}\frac{b_\nu}{t^{n-1}}\Bigg|_{\frac{\sin\frac{\varphi_1}{2}}{b_\nu}}^{2r_{\nu+1}} \le \frac{n}{n-1}\frac{b_\nu^n}{\sin^{n-1}\frac{\varphi_1}{2}} \le$$
$$\le \frac{2}{\sin^{n-1}\frac{\varphi_1}{2}}\Big(\frac{c(r_{\nu+1},0,\infty)}{\psi(r_{\nu+1})}\Big)^n. \tag{56}$$

If $n = 1$, then

$$\sum_{c_q \in G_\nu^+(z)} \frac{\sin\theta_q}{|z-c_q|} \le b_\nu + b_\nu \log\frac{2b_\nu r_{\nu+1}}{\sin\frac{\varphi_1}{2}} \le \frac{c(r_{\nu+1},0,\infty)}{\psi(r_{\nu+1})}\Big(1 + \log\frac{2r_{\nu+1}c(r_{\nu+1},0,\infty)}{\psi(r_{\nu+1})\sin\frac{\varphi_1}{2}}\Big). \tag{57}$$

The second sum in (55) allows the following estimate

$$\sum_{c_q \in G_\nu^-(z)} \frac{\sin\theta_q}{|z-c_q|^n} \le \frac{1}{r^n\sin^n\frac{\varphi_1}{2}} \sum_{c_q \in G_\nu^-(z)} \sin\theta_q \le \frac{c(r_{\nu+1},0,\infty)}{r^n\sin^n\frac{\varphi_1}{2}}. \tag{58}$$

Observe, that $\sin\frac{\varphi_1}{2} = \frac{|\sin\varphi|}{2\cos\frac{\varphi_1}{2}} \ge |\sin\varphi|/2$. Then

Inequalities (56)–(58), the assumptions $\psi(r_{\nu+1}) = O(r_{\nu-1})$ $(\nu \to \infty)$ and $n > 1$ imply that for the light points from G_ν^*, $(r_{\nu-1} \le r_\nu)$ we have

$$\sum_{c_q \in G_\nu} \frac{\sin\theta_q}{|z-c_q|^n} \le K\frac{c^n(r_{\nu+1},0,\infty)}{(\psi(r_{\nu+1})\sin\varphi)^n}\Big(\sin\frac{\varphi_1}{2} + \Big(\frac{\psi(r_{\nu+1})}{r}\Big)^n\Big) \le K\frac{c^n(r_{\nu+1},0,\infty)}{(\psi(r_{\nu+1})\sin\varphi)^n}. \tag{59}$$

Finally, taking into account Lemma 2 and (5) we obtain

$$\sum_{|c_q| \le r_{\nu-2}} \frac{\sin\theta_q}{|z-c_q|^n} \le \sum_{|c_q| \le r_{\nu-2}} \frac{\sin\theta_q}{(r_{\nu-1}-r_{\nu-2})^n} = \frac{c(r_{\nu-2},0,\infty)}{(r_{\nu-1}-r_{\nu-2})^n} \le$$
$$\le \frac{r_{\nu-1}^2 C(r_{\nu-1},0,\infty)}{(r_{\nu-1}-r_{\nu-2})^{n+1}} \le \frac{r_{\nu-1}^2(2S(r_{\nu-1},f)+O(1))}{(r_{\nu-1}-r_{\nu-2})^{n+1}}. \tag{60}$$

The statement of Theorem 1 now follows from (52), (54), (59) and (60). □

Proof of Corollary 1. Let $\gamma > 1$, $r_\nu = \gamma^{\nu/2}$.

Taking into account that $r_{\nu+1}=\gamma r_{\nu-1}\le \gamma r$, $r/\gamma\le r_{\nu-1}$ for $z\in G_\nu$ we obtain

$$|W(z)|\le K\frac{b_\nu^n}{\varepsilon\sin^n\varphi}=\frac{K}{\varepsilon}\Big(\frac{c(\gamma r,0,\infty)}{r^\tau\sin\varphi}\Big)^n,\quad n>1,$$

$$|W(z)|\le \frac{K}{\varepsilon}\frac{c(\gamma r,0,\infty)}{r^\tau}\log\frac{c(\gamma r,0,\infty)}{\sin\varphi},\quad n=1.$$

We estimate the exceptional set for $R\in(r_{m-1},r_m]$, $m\ge\nu_0$,

$$\sum_{|z_{k,\nu}|\le R}\sigma_{k,\nu}\le\sum_{\nu=\nu_0}^{m}12\varepsilon\gamma^{\frac{\tau\nu}{2}}\le 12\varepsilon\frac{\gamma^{\frac{\tau m}{2}}}{1-\gamma^{-\frac{\tau}{2}}}\le\frac{12\varepsilon\gamma^{\frac{\tau}{2}}}{1-\gamma^{-\frac{\tau}{2}}}R^\tau.$$

□

Proof of Corollary 2. Inequalities (12) and (13) can be obtained similar to that in Corollary 1. Further (12) implies (14). It follows from (14) that ($\sum qi_q=n$, $|z|=r$, $z\notin E$),

$$\left|\prod_{q=1}^{n-1}\Big(\frac{d^q\log f(z)}{dz^q}\Big)^{i_q}\right|<K\prod_{q=1}^{n-1}\frac{r^{q(\rho+1+\varepsilon)i_q}}{\sin^{2qi_q}\varphi}=K\frac{r^{(\rho+1+\varepsilon)\sum qi_q}}{(\sin\varphi)^{\sum 2qi_q}}=K\frac{r^{(\rho+1+\varepsilon)n}}{\sin^{2n}\varphi}.\tag{61}$$

An application of Lemma 1 and (61) yield

$$\left|\frac{f^{(n)}(z)}{f(z)}\right|<K\frac{r^{n(\rho+1+\varepsilon)}}{\sin^{2n}\varphi},\ z\notin E,$$

where E satisfies (13). □

3. Examples.

Example 1. Consider the function $f_1(z)=e^{z^\rho}$, $z\in\mathbb{C}_+(1)$, where $\rho>1$, and the branch of the power function is chosen such that $i^\rho=e^{i\pi\rho/2}$. Since f_1 is nonvanishing, $C(r,f_1)=0$, and standard calculations yield $S(r,f_1)\sim K_1(\rho)r^{\rho-1}$, $r\to+\infty$ (cf. Example 2).

Obviously,

$$\left|\frac{d^n\log f_1}{dz^n}\right|=|\rho(\rho-1)\cdots(\rho-n+1)||z|^{\rho-n},\quad \rho\notin\{2,3,\dots,n-1\}.$$

On the other hand, Theorem 1 gives the estimate

$$\left|\frac{d^n\log f_1}{dz^n}\right|\le K\frac{r^{\rho-n}}{\sin^{n+1}\varphi},\quad z=re^{i\varphi}\in\mathbb{C}_+(1).$$

Example 2. Consider the Mittag-Leffler function $E_\rho(z)$, for $\rho\in(\frac12,+\infty)\setminus\{1\}$ (see e.g. [9]). A. A. Gol'dberg proved ([11]) that for all $q>0$ there exists $P>0$ such that

$$\left|\frac{E_\rho'(z)}{E_\rho(z)}\right|\le P|z|^{\rho-1},\quad |z|\to\infty, z\notin\bigcup_k D(\lambda_k,q|\lambda_k|^{1-\rho}),$$

where $\{\lambda_k: k\in\mathbb{N}\}$ is the zero sequence of $E_\rho(z)$ ordered according to increasing moduli. It is known ([9, p.156]) that λ_n satisfies the following asymptotics ($\alpha=1/\rho$)

$$\lambda_n=e^{i\pi\alpha/2}(2\pi n)^\alpha\Big(1-\frac{1}{4\rho^2n}+O_*\Big(\frac{\log n}{n^2}\Big)+i\frac{\log n}{\rho^2 2\pi n}+iO_*\Big(\frac1n\Big)\Big),\quad n\to\infty,\tag{62}$$

$$|\lambda_n|=(2\pi n)^\alpha\Big(1-\frac{1}{4\rho^2n}+O\Big(\frac{\log^2 n}{n^2}\Big)\Big),\quad n\to\infty,\tag{63}$$

where $O_*(\cdot) = O(\cdot)$ stand for real values. The following asymptotics of $E_\rho(z)$ is well-known.

$$E_\rho(z) = \begin{cases} \rho e^{z^\rho} + O\Big(\frac{1}{z}\Big), & |\arg z| \le \frac{\pi}{2\rho}, \\ O\Big(\frac{1}{z}\Big), & \frac{\pi}{2\rho} \le |\arg z| \le \pi, \end{cases} \quad z \to \infty.$$

It allows us to compute its characteristics

$$A(r, E_\rho) \sim \frac{1}{\pi} \int_1^r (t^{-2} - r^{-2}) t^\rho \, dt = \frac{2r^{\rho-1}}{\pi(\rho^2 - 1)}, \quad r \to +\infty, \tag{64}$$

$$B(r, E_\rho) \sim \frac{2}{\pi r} \int_0^{\frac{\pi}{2\rho}} \operatorname{Re}(r^\rho e^{i\rho\theta}) \sin\theta \, d\theta = \frac{2r^{\rho-1}}{\pi} \int_0^{\frac{\pi}{2\rho}} \cos\rho\theta \sin\theta \, d\theta, \quad r \to +\infty. \tag{65}$$

The equality (62) implies $c(r, E_\rho) \sim \frac{\sin\frac{\pi}{2\rho}}{2\pi} r^\rho$, $r \to \infty$. Consequently,

$$C(r, E_\rho) \sim \frac{\sin\frac{\pi}{2\rho}}{\pi} \int_1^r \Big(t^{\rho-2} + \frac{t^\rho}{r^2}\Big) dt = \frac{2\rho \sin\frac{\pi}{2\rho}}{\pi(\rho^2 - 1)} r^{\rho-1}, \quad r \to \infty.$$

So, $S(r, f) \sim K_2 r^{\rho-1}$ as $r \to \infty$, where $K_2 = K_2(\rho)$ is a positive constant.

Following A. A. Gol'dberg, for a given $\eta \in (0, \min\{\pi - \pi\alpha/2, \pi\alpha/2\})$, we consider the angle $W = \{z : \pi\alpha/2 - \eta < \arg z < \pi\alpha/2 + \eta\}$. For $z \in W$ the following asymptotics

$$E_\rho(z) = \rho e^{z^\rho} - \frac{a}{z} + O\Big(\frac{1}{z^2}\Big), \quad a = \frac{1}{\Gamma(1-\alpha)}, \ z \to \infty, \tag{66}$$

and

$$E'_\rho(z) = \rho^2 z^{\rho-1} e^{z^\rho} + O(\frac{1}{z^2}), \quad z \to \infty \tag{67}$$

hold. It follows from (62) that given $q_0 > 0$ there exists $R_0 > 0$ such that for $|\lambda_n| > R_0$ (62) holds and the disks $D(\lambda_n, q|\lambda_n|^{1-\rho})$, $0 < q < q_0$ are pairwise disjoint. Let $n_0 \in \mathbb{N}$ be such that $|\lambda_{n_0}| > R_0$. Let $\{q_n\}$ be a decreasing sequence of positive numbers with $q_n \le q_0$, which will be specified later. At the moment we assume that $\max\{|\lambda_n|^{-2}, |\lambda_n|^{-\rho}\} = o(q_n)$ as $n \to \infty$. For a point $\zeta \in \partial D(\lambda_n, q|\lambda_n|^{1-\rho})$ we have $\zeta = \lambda_n + q|\lambda_n|^{1-\rho} e^{i\theta}$, $\theta \in [0, 2\pi]$, $0 < q \le q_n$. Using the relation ([11])

$$\rho e^{\lambda_n^\rho} = \frac{a}{\lambda_n} + O\Big(\frac{1}{\lambda_n^2}\Big), \quad n \to \infty,$$

and the fact that $|\exp\{\rho q e^{i\theta} + O(\lambda_n^{-\rho})\}| > K_3$ for $n \ge n_0$, $0 < q \le q_n$, $0 \le \theta \le 2\pi$, where $K_3 = K_3(\rho)$ is a positive constant, we deduce

$$\begin{aligned} |E'_\rho(\zeta)| &= \Big|\rho^2 \zeta^{\rho-1} e^{\zeta^\rho} + O\Big(\frac{1}{\zeta^2}\Big)\Big| \ge \\ &= \rho^2 |\zeta|^{\rho-1} |\exp\{\lambda_n^\rho + \rho q e^{i\theta} + O(\lambda_n^{-\rho})\}| + O(|\zeta|^{-2}) = \\ &= \rho^2 |\zeta|^{\rho-1} \Big|\frac{a}{\lambda_n} + O\Big(\frac{1}{\lambda_n^2}\Big)\Big| |\exp\{\rho q e^{i\theta} + O(\lambda_n^{-\rho})\}| + O(|\zeta|^{-2}) \ge \frac{K_4 |\zeta|^{\rho-1}}{|\lambda_n|}, \quad n \to \infty, \end{aligned} \tag{68}$$

where K_4 is a positive constant.

On the other hand, $|\exp\{\rho q e^{i\theta} + O(\lambda_n^{-\rho})\} - 1| = O(q_n)$, $0 < q \le q_n$. A. Gol'dberg proved ([11, p.24]) that

$$E_\rho(\zeta) = \frac{a}{\lambda_n} \Big(\exp\{\rho q e^{i\theta} + O(\lambda_n^{-\rho})\} - 1\Big) + O\Big(\frac{1}{\lambda_n^2}\Big) + O\Big(\frac{1}{\lambda_n^{\rho+1}}\Big).$$

The latter relationships imply $|E_\rho(\zeta)| = O\Big(\frac{q_n}{|\lambda_n|}\Big)$, $0 < q \le q_n$, $\zeta \in \partial D(\lambda_n, q|\lambda_n|^{1-\rho})$ as $n \to \infty$.

Combining this with (68) we obtain

$$\Big|\frac{E'_\rho(\zeta)}{E_\rho(\zeta)}\Big| \ge K_5\frac{|\zeta|^{\rho-1}}{q_n}, \quad \zeta \in D(\lambda_n, q_n|\lambda_n|^{1-\rho}),\ n \to \infty. \tag{69}$$

Let now $\{q_n\}$ be chosen such that $q_n \to 0$ and $\max\{|\lambda_n|^{-2}, |\lambda_n|^{-\rho}\} = o(q_n)$ as $n \to \infty$. Consider the set

$$F = \Big\{\zeta : |\zeta| > R_0, \Big|\frac{E'_\rho(\zeta)}{E_\rho(\zeta)}\Big| \ge K_5\frac{|\zeta|^{\rho-1}}{q_n}, \frac{1}{2}(|\lambda_n| + |\lambda_{n-1}|) \le |\zeta| < \frac{1}{2}(|\lambda_n| + |\lambda_{n+1}|)\Big\}.$$

Then $F \supset \bigcup_{n=n_0}^{\infty} D(\lambda_n, q_n|\lambda_n|^{1-\rho})$, and

$$\sum_{|\lambda_n|\le r} q_n|\lambda|^{1-\rho} = \sum_{n=n_0}^{n(r)} (q_n + o(1))2\pi n^{\frac{1-\rho}{\rho}} \sim 2\pi\rho q_{n(r)}(n(r))^{\frac{1}{\rho}} \sim (2\pi)^{1-\alpha} q_{n(r)} r, \quad r \to \infty.$$

Note that q_n can be tend to zero arbitrarily slow, which shows sharpness of the estimate of exceptional set in Corollary 1 for $\psi(r) = r$.

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