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A. I. Bandura

ENTIRE FUNCTIONS OF BOUNDED INDEX IN FRAME

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We introduce a concept of entire functions having bounded index in a variable direction, i.e. in a frame. An entire function $F\colon\ \mathbb{C}^n\to\mathbb{C}$ is called a function of bounded frame index in a frame $\mathbf{b}(z)$, if there exists $m_0\in\mathbb{Z}_+$ such that for every $m\in\mathbb{Z}_+$ and for all $z\in\mathbb{C}^n$ one has $\dfrac{|\partial^m_{\mathbf{b}(z)}F(z)|}{m!}\le\max\limits_{0\le k\le m_0}\dfrac{|\partial^k_{\mathbf{b}(z)}F(z)|}{k!}$, where $\partial^0_{\mathbf{b}(z)}F(z)=F(z)$, $\partial^1_{\mathbf{b}(z)}F(z)=\sum_{j=1}^n\frac{\partial F}{\partial z_j}(z)\cdot b_j(z)$, $\partial^k_{\mathbf{b}(z)}F(z)=\partial_{\mathbf{b}(z)}(\partial^{k-1}_{\mathbf{b}(z)}F(z))$ for $k\ge 2$ and $\mathbf{b}\colon\ \mathbb{C}^n\to\mathbb{C}^n$ is a entire vector-valued function. There are investigated properties of these functions. We established analogs of propositions known for entire functions of bounded index in direction. The main idea of proof is usage the slice $\{z+t\mathbf{b}(z)\colon\ t\in\mathbb{C}\}$ for given $z\in\mathbb{C}^n$. We proved the following criterion (Theorem 1) describing local behavior of modulus $\partial^k_{\mathbf{b}(z)}F(z+t\mathbf{b}(z))$ on the circle $|t|=\eta$: *An entire function $F\colon\ \mathbb{C}^n\to\mathbb{C}$ is of bounded frame index in the frame $\mathbf{b}(z)$ if and only if for each $\eta>0$ there exist $n_0=n_0(\eta)\in\mathbb{Z}_+$ and $P_1=P_1(\eta)\ge 1$ such that for every $z\in\mathbb{C}^n$ there exists $k_0=k_0(z)\in\mathbb{Z}_+$, $0\le k_0\le n_0$, for which inequality*

$$\max\left\{\left|\partial^{k_0}_{\mathbf{b}(z)}F(z+t\mathbf{b}(z))\right|\colon\ |t|\le\eta\right\}\le P_1\left|\partial^{k_0}_{\mathbf{b}(z)}F(z)\right|$$

holds.

1. Introduction. In recent years, analytic functions of several variables with bounded index have been intensively investigated. The main objects of investigations are such function classes: entire functions of several variables [2, 3, 4, 9, 19, 20], functions analytic in a polydisc [10], in a ball [5, 6, 12] or in the Cartesian product of the complex plane and the unit disc [11].

For entire functions and analytic functions in a ball there were proposed two approaches to introduce a concept of index boundedness in a multidimensional complex space. They generate so-called functions of bounded L-index in a direction, and functions of bounded $\mathbf{L}$-index in joint variables.

Let us introduce some notations and definitions.

Let $\mathbb{R}_+=(0,+\infty)$, $\mathbb{R}^*_+=[0,+\infty)$, $\mathbf{0}=(0,\ldots,0)$, $\mathbf{b}=(b_1,\ldots,b_n)\in\mathbb{C}^n\setminus\{\mathbf{0}\}$ be a given direction, $L\colon\ \mathbb{C}^n\to\mathbb{R}_+$ be a continuous function, $F\colon\ \mathbb{C}^n\to\mathbb{C}$ an entire function. We also put $\partial^0_{\mathbf{b}}F(z)=F(z),\partial_{\mathbf{b}}F(z)=\sum\limits_{j=1}^n\frac{\partial F(z)}{\partial z_j}b_j$, $\partial^k_{\mathbf{b}}F(z)=\partial_{\mathbf{b}}\Big(\partial^{k-1}_{\mathbf{b}}F(z)\Big)$, $k\ge 2$.

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An entire function $F\colon\ \mathbb{C}^n \to \mathbb{C}$ is called ([2, 3, 4]) *a function of bounded L-index in a direction* $\mathbf{b}$, if there exists $m_0 \in \mathbb{Z}_+$ such that for every $m \in \mathbb{Z}_+$ and for all $z \in \mathbb{C}^n$ one has

$$\frac{|\partial_{\mathbf{b}}^m F(z)|}{m! L^m(z)} \le \max\Big\{\frac{|\partial_{\mathbf{b}}^k F(z)|}{k!} : 0 \le k \le m_0\Big\}. \tag{1}$$

One should observe that

$$\partial_{\mathbf{b}}^k F(z) = \frac{k!}{2\pi i} \int\limits_{|\tau|=r} \frac{F(z+\tau\mathbf{b})}{\tau^{k+1}} d\tau \quad \forall z \in \mathbb{C}^n. \tag{2}$$

In our previous investigations we consider that $\mathbf{b}$ is a constant vector, i.e. $\mathbf{b}$ does not depend on variable $z \in \mathbb{C}^n$. Now, let us suppose that $\mathbf{b} = \mathbf{b}(z) = (b_1(z), \ldots, b_n(z))$ is some entire vector-valued function of variable z.

Definition 1. *An entire function $F\colon\ \mathbb{C}^n \to \mathbb{C}$ is called a function of bounded index in a frame $\mathbf{b}(z)$, if there exists $m_0 \in \mathbb{Z}_+$ such that for every $m \in \mathbb{Z}_+$ and for all $z \in \mathbb{C}^n$ one has*

$$\frac{|\partial_{\mathbf{b}(z)}^m F(z)|}{m!} \le \max\Big\{\frac{|\partial_{\mathbf{b}(z)}^k F(z)|}{k!} : 0 \le k \le m_0\Big\}, \tag{3}$$

where

$$\partial_{\mathbf{b}(z)}^0 F(z) = F(z), \quad \partial_{\mathbf{b}(z)}^1 F(z) = \sum_{j=1}^n \frac{\partial F}{\partial z_j}(z) \cdot b_j(z), \quad \partial_{\mathbf{b}(z)}^k F(z) = \partial_{\mathbf{b}(z)}(\partial_{\mathbf{b}(z)}^{k-1} F(z))$$

for $k \ge 2$.

We denote $\partial_{\mathbf{b}(z)} F(z) \equiv \partial_{\mathbf{b}(z)}^1 F(z)$.

The least such integer number m_0, obeying (1), is called index in the frame $\mathbf{b}(z)$ of the function $F(z)$ and is denoted by $N_{\mathbf{b}}(F)$. If such m_0 does not exist, then we put $N_{\mathbf{b}}(F) = \infty$, and the function F is said to be of unbounded index in the frame $\mathbf{b}$ in this case. If $\mathbf{b}(z) \equiv (b_1, \ldots, b_n)$ is a constant vector then we obtain usual definition of bounded index in the direction $\mathbf{b}$.

If $n = 1$ and $\mathbf{b}(z) \equiv 1$, then the definition matches with the definition of function of bounded index introduces by B. Lepson [17] (see also [18]). In this case, $N(f)\colon\ = N_1(f)$.

The notion of boundedness of the L-index in direction uses the restriction of the function to the slices $\{z^0 + t\mathbf{b}\colon\ t \in \mathbb{C}\}$. For fixed $\mathbf{b} \in \mathbb{C}^n \setminus \{0\}$ and $z^0 \in \mathbb{C}^n$, using considerations from the one-dimensional case, we obtain the estimates which are uniform in $z^0 \in \mathbb{C}^n$. This is a short description of the method.

2. Sufficient Sets. Now we prove several propositions describing a connection between functions of bounded index in direction and functions of bounded index of one variable. The similar results for entire functions of several variables were obtained in [2, 3, 7], for slice holomorphic functions in [8] . The next proofs use ideas from the mentioned papers. Denote $g_z(t) = F(z + t\mathbf{b}(z))$.

Proposition 1. *Let $\mathbf{b}\colon\ \mathbb{C}^n \to \mathbb{C}^n$ be an entire vector-valued function. If an entire function $F\colon\ \mathbb{C}^n \to \mathbb{C}$ has bounded frame index in the frame $\mathbf{b}(z)$ then for every $z^0 \in \mathbb{C}^n$ the entire function $g_{z^0}(t)$ is of bounded index and $N(g_{z^0}) \le N_{\mathbf{b}(z)}(F)$.*

Proof. Let $z^0 \in \mathbb{C}^n$. Denote $g_{z^0}(t) = F(z^0 + t\mathbf{b}(z^0))$. As for all $p \in \mathbb{N}$ one has

$$g'_{z^0}(t) = \sum_{j=1}^{n} \frac{\partial F(z^0 + t\mathbf{b}(z^0))}{\partial z_j} \cdot b_j(z^0) = \partial_{\mathbf{b}(z^0)} F(z^0 + t\mathbf{b}(z^0)), \tag{4}$$

$$g^{(p)}_{z^0}(t) = \sum_{j=1}^{n} \frac{\partial(\partial^{p-1}_{\mathbf{b}(z^0)} F(z^0 + t\mathbf{b}(z^0)))}{\partial z_j} \cdot b_j(z^0) = \partial^{p}_{\mathbf{b}(z^0)} F(z^0 + t\mathbf{b}(z^0)), \tag{5}$$

then by the definition of bounded index in the frame $\mathbf{b}(z)$ for all $t \in \mathbb{C}$ and $p \in \mathbb{Z}_+$ we obtain

$$\frac{|g^{(p)}_{z^0}(t)|}{p!} = \frac{|\partial^{p}_{\mathbf{b}(z^0)} F(z^0 + t\mathbf{b}(z^0))|}{p!} \le \max\left\{ \frac{|\partial^{k}_{\mathbf{b}(z^0)} F(z^0 + t\mathbf{b}(z^0))|}{k!} \colon\ 0 \le k \le N_{\mathbf{b}}(F) \right\} =$$

$$= \max\left\{ \frac{|g^{(k)}_{z^0}(t)|}{k!} \colon\ 0 \le k \le N_{\mathbf{b}}(F) \right\}.$$

Hence, we obtain that $g_z(t)$ is of bounded index and $N(g) \le N_{\mathbf{b}}(F)$. Proposition 1 is proved. □

Remark 1. Treating $F(z + t\mathbf{b}(z))$ as entire function of variable t for given z and applying Cauchy's formula, we obtain

$$\partial^{k}_{\mathbf{b}(z)} F(z) = g^{(k)}_{z}(0) = \frac{k!}{2\pi i} \int\limits_{|\tau|=r} \frac{g_z(\tau)}{\tau^{k+1}} d\tau = \frac{k!}{2\pi i} \int\limits_{|\tau|=r} \frac{F(z + \tau\mathbf{b}(z))}{\tau^{k+1}} d\tau. \tag{6}$$

Equality (5) implies that the proposition holds.

Proposition 2. *Let* $\mathbf{b}\colon\ \mathbb{C}^n \to \mathbb{C}^n$ *be an entire vector-valued function. If an entire function* $F\colon\ \mathbb{C} \to \mathbb{C}$ *has bounded index in the frame* $\mathbf{b}(z)$ *then*

$$N_{\mathbf{b}}(F) = \max\left\{ N(g_{z^0})\colon\ z^0 \in \mathbb{C}^n \right\}.$$

Proposition 3. *Let* $\mathbf{b}\colon\ \mathbb{C}^n \to \mathbb{C}^n$ *be an entire vector-valued function. An entire function* $F\colon\ \mathbb{C}^n \to \mathbb{C}$ *has bounded index in the frame* $\mathbf{b}$ *if and only if there exists a number* $M > 0$ *such that for all* $z^0 \in \mathbb{C}^n$ *the function* $g_{z^0}(t)$ *is of bounded index with* $N(g_{z^0}) \le M < +\infty$*, as a function of variable* $t \in \mathbb{C}$*. Thus,* $N_{\mathbf{b}}(F) = \max\{N(g_{z^0})\colon\ z^0 \in \mathbb{C}^n\}$*.*

Proof. The *necessity* follows from Proposition 1.
Sufficiency. Since $N(g_{z^0}) \le M$, there exists $\max\{N(g_{z^0})\colon\ z^0 \in \mathbb{C}^n\}$. We denote $N_{\mathbf{b}}(F) = \max\{N(g_{z^0})\colon\ z^0 \in \mathbb{C}^n\} < +\infty$. Suppose that $N_{\mathbf{b}}(F)$ is not the index in the frame $\mathbf{b}$ of the function $F(z)$. It means that there exists $n^* > N_{\mathbf{b}}(F)$ and $z^* \in \mathbb{C}^n$ such that

$$\frac{|\partial^{n^*}_{\mathbf{b}(z^*)} F(z^*)|}{n^*!} > \max\left\{ \frac{|\partial^{k}_{\mathbf{b}(z^*)} F(z^*)|}{k!} \colon\ 0 \le k \le N_{\mathbf{b}}(F) \right\}. \tag{7}$$

Since for $g_{z^0}(t) = F(z^0 + t\mathbf{b}(z^0))$ we have $g^{(p)}_{z^0}(t) = \partial^{p}_{\mathbf{b}(z^0)} F(z^0 + t\mathbf{b}(z^0))$, inequality (7) can be rewritten as

$$\frac{\left|g^{(n^*)}_{z^*}(0)\right|}{n^*!} > \max\left\{ \frac{|g^{(k)}_{z^*}(0)|}{k!} \colon\ 0 \le k \le N_{\mathbf{b}}(F) \right\},$$

but it is impossible (it contradicts that all indices $N(g_{z_0})$ are not greater than $N_{\mathbf{b}}(F)$). Therefore $N_{\mathbf{b}}(F, L)$ is the index in the frame $\mathbf{b}(z)$ of the function $F(z)$. Proposition 3 is proved. □

3. Local Behavior of functions of bounded index in frame. The following proposition is crucial in theory of functions of bounded index. It initializes series of propositions which are necessary to prove logarithmic criterion of index boundedness. It was first obtained by G. H. Fricke [15] for entire functions of bounded index. Later the proposition was generalized for entire functions of bounded l-index [16], analytic functions of bounded l-index [21], entire functions of bounded L-index in direction [2], functions analytic in the unit ball with bounded L-index in direction [6], functions analytic in a polydisc [10] or in a ball [12] with bounded $\mathbf{L}$-index in joint variables and for slice holomorphic functions [8].

Theorem 1. *Let* $\mathbf{b}\colon\ \mathbb{C}^n \to \mathbb{C}^n$ *be an entire vector-valued function. An entire function* $F\colon\ \mathbb{C}^n \to \mathbb{C}$ *is of bounded frame index in the frame* $\mathbf{b}(z)$ *if and only if for each* $\eta > 0$ *there exist* $n_0 = n_0(\eta) \in \mathbb{Z}_+$ *and* $P_1 = P_1(\eta) \geq 1$ *such that for every* $z \in \mathbb{C}^n$ *there exists* $k_0 = k_0(z) \in \mathbb{Z}_+$, $0 \leq k_0 \leq n_0$, *for which inequality*

$$\max\left\{\left|\partial^{k_0}_{\mathbf{b}(z)}F(z+t\mathbf{b}(z))\right|:\ |t|\leq\eta\right\}\leq P_1\left|\partial^{k_0}_{\mathbf{b}(z)}F(z)\right| \tag{8}$$

holds.

Proof. Our proof is based on the proof of an appropriate theorem for entire functions of bounded L-index in direction ([2], [3, p.20],[4, p.88-89]).
Necessity. Let $N_{\mathbf{b}}(F) \equiv N < +\infty$. Let $[a]$, $a \in \mathbb{R}$, stands for the integer part of the number a in this proof. We denote

$$q(\eta) = [2\eta(N+1)] + 1.$$

For $z \in \mathbb{C}^n$ and $p \in \{0, 1, \ldots, q(\eta)\}$ we put

$$R^{\mathbf{b}}_p(z,\eta) = \max\left\{\frac{|\partial^k_{\mathbf{b}(z)}F(z+t\mathbf{b}(z))|}{k!}:\ |t|\leq\frac{p\eta}{q(\eta)}, 0\leq k\leq N\right\}.$$

Note that $|t| \leq \frac{p\eta}{q(\eta)} \leq \eta$. It is clear that $R^{\mathbf{b}}_p(z,\eta)$ is well-defined.
Let $k^z_p \in \mathbb{Z}$, $0 \leq k^z_p \leq N$, and $t^z_p \in \mathbb{C}$, $|t^z_p| \leq \frac{p\eta}{q(\eta)}$, be such that

$$R^{\mathbf{b}}_p(z,\eta) = \frac{|\partial^{k^z_p}_{\mathbf{b}(z)}F(z+t^z_p\mathbf{b}(z))|}{k^z_p!}. \tag{9}$$

However, for every given $z \in \mathbb{C}^n$ the function $F(z+t\mathbf{b}(z))$ and $\partial^k_{\mathbf{b}(z)}F(z+t\mathbf{b}(z))$ are entire functions of the variable t. Then by the maximum modulus principle, equality (9) holds for t^z_p such that $|t^z_p| = \frac{p\eta}{q(\eta)}$. We set $\widetilde{t^z_p} = \frac{p-1}{p}t^z_p$. Then

$$|\widetilde{t^z_p}| = \frac{(p-1)\eta}{q(\eta)}, \tag{10}$$

$$|\widetilde{t^z_p} - t^z_p| = \frac{|t^z_p|}{p} = \frac{\eta}{q(\eta)}. \tag{11}$$

It follows from (10) and the definition of $R^{\mathbf{b}}_{p-1}(z,\eta)$ that

$$R^{\mathbf{b}}_{p-1}(z,\eta) \geq \frac{|\partial^{k^z_p}_{\mathbf{b}(z)}F(z+\widetilde{t^z_p}\mathbf{b}(z))|}{k^z_p!}.$$

Therefore,

$$0 \le R^{\mathbf{b}}_p(z,\eta) - R^{\mathbf{b}}_{p-1}(z,\eta) \le \frac{\left|\partial^{k_p^z}_{\mathbf{b}(z)} F(z + t_p^z \mathbf{b}(z))\right| - \left|\partial^{k_p^z}_{\mathbf{b}(z)} F(z + \widetilde{t}_p^z \mathbf{b}(z))\right|}{k_p^z!} =$$

$$= \frac{1}{k_p^z!} \int\limits_0^1 \frac{d}{ds} \left|\partial^{k_p^z}_{\mathbf{b}(z)} F(z + (\widetilde{t}_p^z + s(t_p^z - \widetilde{t}_p^z))\mathbf{b}(z))\right| ds. \tag{12}$$

For every analytic complex-valued function of real variable $\varphi(s)$, $s \in \mathbb{R}$, the inequality $\frac{d}{ds}|\varphi(s)| \le \left|\frac{d}{ds}\varphi(s)\right|$ holds, where $\varphi(s) \ne 0$. Applying this inequality to (12) and using the mean value theorem we obtain

$$R^{\mathbf{b}}_p(z,t_0,\eta) - R^{\mathbf{b}}_{p-1}(z,t_0,\eta) \le$$

$$\le \frac{1}{k_p^z!} \int\limits_0^1 \left|\frac{d}{ds} \partial^{k_p^z}_{\mathbf{b}(z)} F(z + (\widetilde{t}_p^z + s(t_p^z - \widetilde{t}_p^z))\mathbf{b}(z))\right| ds =$$

$$= \frac{|t_p^z - \widetilde{t}_p^z|}{k_p^z!} \int\limits_0^1 \left|\partial^{k_p^z+1}_{\mathbf{b}(z)} F(z + (\widetilde{t}_p^z + s(t_p^z - \widetilde{t}_p^z))\mathbf{b}(z))\right| ds =$$

$$= \frac{|t_p^z - \widetilde{t}_p^z|}{k_p^z!} \left|\partial^{k_p^z+1}_{\mathbf{b}(z)} F(z + (\widetilde{t}_p^z + s^*(t_p^z - \widetilde{t}_p^z))\mathbf{b}(z))\right| =$$

$$= (k_p^z+1)|t_p^z - \widetilde{t}_p^z| \frac{|\partial^{k_p^z+1}_{\mathbf{b}(z)} F(z + (\widetilde{t}_p^z + s^*(t_p^z - \widetilde{t}_p^z))\mathbf{b}(z))|}{(k_p^z+1)!}, \tag{13}$$

where $s^* \in [0,1]$. The point $\widetilde{t}_p^z + s^*(t_p^z - \widetilde{t}_p^z)$ belongs to the set

$$\left\{ t \in \mathbb{C} \colon\ |t| \le \frac{p\eta}{q(\eta)} \le \eta \right\}.$$

Using the definition of index boundedness in frame, the definition of $q(\eta)$, inequalities (11) and (13), for $k_p^z \le N$ we have

$$R^{\mathbf{b}}_p(z,\eta) - R^{\mathbf{b}}_{p-1}(z,\eta) \le \frac{|\partial^{k_p^z+1}_{\mathbf{b}(z)} F(z + (\widetilde{t}_p^z + s^*(t_p^z - \widetilde{t}_p^z))\mathbf{b}(z))|}{(k_p^z+1)!} (k_p^z+1)|t_p^z - \widetilde{t}_p^z| \le \eta \frac{N+1}{q(\eta)} \times$$

$$\times \max\left\{ \frac{|\partial^{k}_{\mathbf{b}(z)} F(z + (\widetilde{t}_p^z + s^*(t_p^z - \widetilde{t}_p^z))\mathbf{b}(z))|}{k!} \colon\ 0 \le k \le N \right\} \le \eta \frac{N+1}{q(\eta)} R^{\mathbf{b}}_p(z,\eta) \le$$

$$\le \frac{\eta(N+1)}{[2\eta(N+1)]+1} R^{\mathbf{b}}_p(z,\eta) \le \frac{1}{2} R^{\mathbf{b}}_p(z,\eta)$$

It follows that $R^{\mathbf{b}}_p(z,\eta) \le 2R^{\mathbf{b}}_{p-1}(z,\eta)$.
Hence,

$$\max\left\{ \frac{|\partial^{k}_{\mathbf{b}(z)} F(z + t\mathbf{b}(z))|}{k!} \colon\ |t| \le \eta, 0 \le k \le N \right\} = R^{\mathbf{b}}_{q(\eta)}(z,\eta) \le$$

$$\le 2R^{\mathbf{b}}_{q(\eta)-1}(z,\eta) \le 2^2 R^{\mathbf{b}}_{q(\eta)-2}(z,\eta) \le$$
$$\le \dots \le 2^{q(\eta)} R^{\mathbf{b}}_0(z,\eta) =$$
$$= 2^{q(\eta)} \max\left\{ \frac{|\partial^k_{\mathbf{b}(z)} F(z)|}{k!}\colon\ 0 \le k \le N \right\}. \tag{14}$$

Let $k_z \in \mathbb{Z}$, $0 \le k_z \le N$, and $\widetilde{t_z} \in \mathbb{C}$, $|\widetilde{t_z}| = \frac{\eta}{L(z)}$ be such that

$$\frac{|\partial^{k_z}_{\mathbf{b}(z)} F(z)|}{k_z!} = \max_{0\le k\le N} \frac{|\partial^{k}_{\mathbf{b}(z)} F(z)|}{k!},$$

and

$$|\partial^{k_z}_{\mathbf{b}(z)} F(z+\widetilde{t_z}\mathbf{b}(z))| = \max\{|\partial^{k_z}_{\mathbf{b}(z)} F(z+t\mathbf{b}(z))|\colon\ |t|\le\eta\}.$$

Inequality (14) implies

$$\frac{|\partial^{k_z}_{\mathbf{b}(z)} F(z+\widetilde{t_z}\mathbf{b}(z))|}{k_z!} = \max\left\{\frac{|\partial^{k_z}_{\mathbf{b}(z)} F(z+t\mathbf{b}(z))|}{k_z!}\colon\ |t|\le\eta\right\} \le$$
$$\le \max\left\{\frac{|\partial^{k}_{\mathbf{b}(z)} F(z+t\mathbf{b}(z))|}{k!}\colon\ |t|\le\eta, 0\le k\le N\right\} \le$$
$$\le (2)^{q(\eta)} \frac{\left|\partial^{k_z}_{\mathbf{b}(z)} F(z)\right|}{k_z!}.$$

Hence,

$$\max\left\{|\partial^{k_z}_{\mathbf{b}(z)} F(z+t\mathbf{b}(z))|\colon\ |t|\le\eta\right\} \le 2^{q(\eta)} |\partial^{k_z}_{\mathbf{b}(z)} F(z)|.$$

Thus, we obtain (8) with $n_0 = N_{\mathbf{b}}(F)$ and

$$P_1(\eta) = 2^{q(\eta)}.$$

Necessity is proved.

Sufficiency. Suppose that for each $\eta > 0$ there exist $n_0 = n_0(\eta) \in \mathbb{Z}_+$ and $P_1 = P_1(\eta) \ge 1$ such that for every $z \in \mathbb{C}^n$ there exists $k_0 = k_0(z) \in \mathbb{Z}_+$, $0 \le k_0 \le n_0$, for which inequality (8) holds. We choose $\eta > 1$ and $j_0 \in \mathbb{N}$ such that $P_1 \le \eta^{j_0}$. For given $z \in \mathbb{C}^n$, $k_0 = k_0(z)$ and $j \ge j_0$ by Cauchy's formula for $F(z+t\mathbf{b}(z))$ as a function of one variable t

$$\partial^{k_0+j}_{\mathbf{b}(z)} F(z) = \frac{j!}{2\pi i} \int\limits_{|t|=\eta} \frac{\partial^{k_0}_{\mathbf{b}(z)} F(z+t\mathbf{b}(z))}{t^{j+1}} dt.$$

Therefore, in view of (8) we have

$$\frac{|\partial^{k_0+j}_{\mathbf{b}(z)} F(z)|}{j!} \le \frac{1}{2\pi} \int\limits_{|t|=\eta} \frac{|\partial^{k_0}_{\mathbf{b}(z)} F(z+t\mathbf{b}(z))|}{|t|^{j+1}} |dt| \le$$
$$\le \frac{1}{\eta^j} \max\left\{|\partial^{k_0}_{\mathbf{b}(z)} F(z+t\mathbf{b}(z))|\colon\ |t|=\eta\right\} \le \frac{P_1}{\eta^j} |\partial^{k_0}_{\mathbf{b}(z)} F(z)|,$$

Hence, for all $j \geq j_0$, $z \in \mathbb{C}^n$

$$\frac{|\partial^{k_0+j}_{\mathbf{b}(z)}F(z)|}{(k_0+j)!} \leq \frac{j!k_0!}{(j+k_0)!}\frac{P_1}{\eta^j}\frac{|\partial^{k_0}_{\mathbf{b}(z)}F(z)|}{k_0!} \leq \eta^{j_0-j}\frac{|\partial^{k_0}_{\mathbf{b}(z)}F(z)|}{k_0!} \leq \frac{|\partial^{k_0}_{\mathbf{b}(z)}F(z)|}{k_0!}.$$

Here we used that $\frac{j!k_0!}{(j+k_0)!} \leq 1$. Since $k_0 \leq n_0$, the numbers $n_0 = n_0(\eta)$ and $j_0 = j_0(\eta)$ are independent of z and t_0, this inequality means that the function F has bounded index in the frame $\mathbf{b}(z)$ and $N_{\mathbf{b}}(F) \leq n_0 + j_0$. The proof of Theorem 1 is complete. □

The following assertion is an analog of propositions established for the L-index in direction [2, 6]. It is a consequence of the previous theorem.

Corollary 1. *Let $\mathbf{b}\colon \mathbb{C}^n \to \mathbb{C}^n$ be an entire vector-valued function. An entire function $F\colon \mathbb{C}^n \to \mathbb{C}$ is of bounded index in the frame $\mathbf{b}(z)$ if and only if F is of bounded index in the frame $\alpha\mathbf{b}(z)$, where $\alpha \in \mathbb{C}$.*

Proposition 4. *Let $\mathbf{b}\colon \mathbb{C}^n \to \mathbb{C}^n$ be an entire vector-valued function, $\alpha : \mathbb{C}^n \to \mathbb{C}\setminus\{0\}$ be an entire function and an entire function $F\colon \mathbb{C}^n \to \mathbb{C}$ be of bounded index in the frame $\mathbf{b}(z)$. If either $\alpha(z) = \exp(\sum_{j=1}^n z_j)$ or $N_{\mathbf{b}}(F) \in \{0,1\}$, is true then F is of bounded index in the frame $\alpha(z)\mathbf{b}(z)$.*

Proof. In the case $\alpha(z) = \exp(\sum_{j=1}^n z_j)$ we have $\frac{\partial\alpha(z)}{\partial z_j} = \alpha(z)$ for $j \in \{1,\dots,n\}$. Therefore, $\partial^k_{\alpha\mathbf{b}(z)}F(z) = \alpha^k\partial^k_{\mathbf{b}(z)}F(z)$, and we can repeat considerations from previous proof, replacing α by $\alpha(z)$.

If $N_{\mathbf{b}}(F) \in \{0,1\}$, then in Theorem 1 inequality (8) holds with $n_0 \in \{0,1\}$ (see its proof). But for such $n_0 = 0$ one has

$$\partial^0_{\alpha\mathbf{b}(z)}F(z) = \alpha^0\partial^0_{\mathbf{b}(z)}F(z) = F(z)$$

and for $n_0 = 1$ one has

$$\partial^1_{\alpha\mathbf{b}(z)}F(z) = \sum_{j=1}^n \frac{\partial F}{\partial z_j}(z)\cdot\alpha(z)\cdot b_j(z) = \alpha\sum_{j=1}^n \frac{\partial F}{\partial z_j}(z)\cdot b_j(z) = \alpha\partial^1_{\mathbf{b}(z)}F(z).$$

Hence, by Theorem 1 $(\forall\eta > 0)$ and for $n_0(\eta) \in \{0,1\}$ $(\exists P_1(\eta) \geq 1)$ $(\forall z \in \mathbb{C}^n)$ $(\exists k_0 = k_0(z) \in \mathbb{Z}_+,\ 0 \leq k_0 \leq n_0)$, and the following inequality is valid

$$\max\Big\{|\partial^{k_0}_{\mathbf{b}(z)}F(z+t\mathbf{b}(z))|\colon\ |t| \leq \eta\Big\} \leq P_1|\partial^{k_0}_{\mathbf{b}(z)}F(z)|. \tag{15}$$

Since $\partial^k_{\alpha(z)\mathbf{b}(z)}F(z) = \alpha^k(z)\cdot\partial^k_{\mathbf{b}(z)}F(z)$ for $k \in \{0,1\}$, inequality (15) is equivalent to the inequality

$$\max\Big\{|\alpha(z)|^{k_0}|\partial^{k_0}_{\mathbf{b}(z)}F(z+t\mathbf{b}(z))|\colon\ |t| \leq \eta\Big\} \leq P_1|\alpha(z)|^{k_0}|\partial^{k_0}_{\mathbf{b}(z)}F(z)|$$

as well as to the inequality

$$\max\bigg\{\Big|\partial^{k_0}_{\alpha(z)\mathbf{b}(z)}F\Big(z+\frac{t}{\alpha(z)}\alpha(z)\mathbf{b}(z)\Big)\Big|\colon\ |t/\alpha(z)| \leq \eta/|\alpha(z)|\bigg\} \leq P_1|\partial^{k_0}_{\alpha\mathbf{b}(z)}F(z)|.$$

Denoting $t^* = \frac{t}{\alpha(z)}$, $\eta^* = \frac{\eta}{|\alpha(z)|}$, we obtain

$$\max\Big\{|\partial^{k_0}_{\alpha(z)\mathbf{b}(z)}F(z+t^*\alpha(z)\mathbf{b}(z))|\colon\ |t^*| \leq \eta^*\Big\} \leq P_1|\partial^{k_0}_{\alpha(z)\mathbf{b}(z)}F(z)|.$$

By Theorem 1 the function $F(z)$ is of bounded index in the frame $\alpha(z)\mathbf{b}(z)$. □

Using Fricke's idea [14], we obtain a modification of Theorem 1.

Theorem 2. *Let $F\colon \mathbb{C}^n \to \mathbb{C}$ be an entire function, $\mathbf{b}\colon \mathbb{C}^n \to \mathbb{C}^n$ be a entire vector-valued function. If there exist $\eta > 0$, $n_0 = n_0(\eta) \in \mathbb{Z}_+$ and $P_1 = P_1(\eta) \geq 1$ such that for all $z \in \mathbb{C}^n$ there exists $k_0 = k_0(z) \in \mathbb{Z}_+$, $0 \leq k_0 \leq n_0$, for which the inequality holds*

$$\max\{|\partial^{k_0}_{\mathbf{b}(z)} F(z + t\mathbf{b}(z))|\colon\ |t| \leq \eta\} \leq P_1 |\partial^{k_0}_{\mathbf{b}(z)} F(z)|,$$

then the function F has bounded index in the frame $\mathbf{b}(z)$.

Proof. Our proof is based on the proof of appropriate theorem for entire functions of bounded L-index in direction [13].

Assume that there exist $\eta > 0$, $n_0 = n_0(\eta) \in \mathbb{Z}_+$ and $P_1 \geq 1$ such that for every $z \in \mathbb{C}^n$ there exists $k_0 = k_0(z) \in \mathbb{Z}_+$, $0 \leq k_0 \leq n_0$, for which

$$\max\{|\partial^{k_0}_{\mathbf{b}(z)} F(z + t\mathbf{b}(z))|\colon\ |t| \leq \eta\} \leq P_1 |\partial^{k_0}_{\mathbf{b}(z)} F(z)|. \tag{16}$$

If $\eta > 1$, then we choose $j_0 \in \mathbb{N}$ such that $P_1 \leq \eta^{j_0}$. And for $\eta \in (0; 1]$ we choose $j_0 \in \mathbb{N}$ obeying the inequality $\frac{j_0! k_0!}{(j_0+k_0)!} P_1 \leq 1$. This j_0 exists because

$$\frac{j_0! k_0!}{(j_0 + k_0)!} P_1 = \frac{k_0!}{(j_0 + 1)(j_0 + 2) \cdot \ldots \cdot (j_0 + k_0)} P_1 \to 0,\ j_0 \to \infty.$$

Applying Cauchy's formula to the function $F(z+t\mathbf{b}(z))$ as a function of complex variable t for $j \geq j_0$ we obtain that for every $z \in \mathbb{C}^n$ there exists integer $k_0 = k_0(z)$, $0 \leq k_0 \leq n_0$, and

$$\partial^{k_0+j}_{\mathbf{b}(z)} F(z) = \frac{j!}{2\pi i} \int\limits_{|t|=\eta} \frac{\partial^{k_0}_{\mathbf{b}(z)} F(z + t\mathbf{b}(z))}{t^{j+1}} dt.$$

Taking into account (16), one has

$$\frac{|\partial^{k_0+j}_{\mathbf{b}(z)} F(z)|}{j!} \leq \frac{1}{\eta^j} \max\left\{|\partial^{k_0}_{\mathbf{b}(z)} F(z + t\mathbf{b}(z))|\colon\quad |t| = \eta\right\} \leq \frac{P_1}{\eta^j} |\partial^{k_0}_{\mathbf{b}(z)} F(z)|. \tag{17}$$

In view of the choice of j_0 for $\eta > 1$ and for all $j \geq j_0$ we deduce

$$\frac{|\partial^{k_0+j}_{\mathbf{b}(z)} F(z)|}{(k_0 + j)!} \leq \frac{j! k_0!}{(j + k_0)!} \frac{P_1}{\eta^j} \frac{|\partial^{k_0}_{\mathbf{b}(z)} F(z)|}{k_0!} \leq \eta^{j_0 - j} \frac{|\partial^{k_0}_{\mathbf{b}(z)} F(z)|}{k_0!} \leq \frac{|\partial^{k_0}_{\mathbf{b}(z)} F(z)|}{k_0!}.$$

Since $k_0 \leq n_0$, the numbers $n_0 = n_0(\eta)$ and $j_0 = j_0(\eta)$ are independent of z, and $z \in \mathbb{C}^n$ is arbitrary, the last inequality means that the function F is of bounded index in the frame $\mathbf{b}(z)$ and $N_{\mathbf{b}}(F) \leq n_0 + j_0$.

If $\eta \in (0, 1)$, then (17) implies for all $j \geq j_0$

$$\frac{|\partial^{k_0+j}_{\mathbf{b}(z)} F(z)|}{(k_0 + j)!} \leq \frac{j! k_0! P_1}{(j + k_0)!} \frac{|\partial^{k_0}_{\mathbf{b}(z)} F(z)|}{\eta^j k_0!} \leq \frac{|\partial^{k_0}_{\mathbf{b}(z)} F(z)|}{\eta^j k_0!}$$

or in view of the choice of j_0

$$\frac{|\partial^{k_0+j}_{\mathbf{b}(z)} F(z)|}{(k_0 + j)!} \eta^{k_0+j} \leq \frac{|\partial^{k_0}_{\mathbf{b}(z)} F(z)|}{k_0!} \eta^{k_0}.$$

Since $\partial^p_{\eta\mathbf{b}(z)}F(z) = \eta^p\partial^p_{\mathbf{b}(z)}F(z)$, one has

$$\frac{|\partial^{k_0+j}_{\eta\mathbf{b}(z)}F(z)|}{(k_0+j)!} \leq \frac{|\partial^{k_0}_{\eta\mathbf{b}(z)}F(z)|}{k_0!}.$$

Thus, the function F has bounded index in the frame $\eta\mathbf{b}$. Then by Theorem 1 the function F is of bounded index in the frame $\mathbf{b}(z)$. Theorem is proved. □

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Ivano-Frankivsk National Technical University of Oil and Gas
Ivano-Frankivsk, Ukraine
andriykopanytsia@gmail.com

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