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# THE BOUNDEDNESS OF A CLASS OF SEMICLASSICAL FOURIER INTEGRAL OPERATORS ON SOBOLEV SPACE $H^s$

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We introduce the relevant background information that will be used throughout the paper. Following that, we will go over some fundamental concepts from the theory of a particular class of semiclassical Fourier integral operators (symbols and phase functions), which will serve as the starting point for our main goal.

Furthermore, these integral operators turn out to be bounded on $S(\mathbb{R}^n)$ the space of rapidly decreasing functions (or Schwartz space) and its dual $S'(\mathbb{R}^n)$ the space of temperate distributions.

Moreover, we will give a brief introduction about $H^s(\mathbb{R}^n)$ Sobolev space (with $s \in \mathbb{R}$). Results about the composition of semiclassical Fourier integral operators with its $L^2$-adjoint are proved. These allow to obtain results about the boundedness on the Sobolev spaces $H^s(\mathbb{R}^n)$.

**1. Introduction.** A semiclassical Fourier integral operator or $h$-FIO for short has the following form

$$(I(a,\varphi;h)u)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{i}{h}\varphi(x,\xi,y)} a(x,\xi,y)\, u(y)\, dyd\xi,\ \ u \in S(\mathbb{R}^n),$$

where $S(\mathbb{R}^n)$ of rapidly decreasing functions (or Schwartz space), $\varphi$ is called the phase function and $a$ is the amplitude of the FIO $I(a,\varphi;h)$; $h$ is a semiclassical parameter $h \in ]0,h_0]$, $h_0 > 0$. In particular when $\varphi(x,\xi,y) = \langle x-y,\xi\rangle$,

$$I(a,\varphi;h) := Op_h(a) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y,\xi\rangle} a(x,\xi,y)\, u(y)\, dyd\xi$$

is called a $h$-pseudodifferential operator see [13, 11, 14].

Historically, a systematic study of smooth FIO with amplitudes in $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that

$$\left|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)\right| \le C_{\alpha,\beta}(1+|\xi|)^{m-\rho|\alpha|+\delta|\beta|}$$

(i.e. $a(x,\xi) \in S^m_{\rho,\delta}$), and phase functions in $C^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus 0)$ homogenous of degree 1 in the frequency variable $\xi$ and with non-vanishing determinant of the mixed Hessian matrix (i.e. non-degenerate phase functions), was initiated in the classical paper of L. Hörmander [10]. Furthermore, G. Eskin [6] (in the case $a \in S^0_{1,0}$) and Hörmander [10] (in the case $a \in S^0_{\rho,1-\rho}$, $\frac{1}{2} < \rho \le 1$) showed the local $L^2$ boundedness of FIO with non-degenerate phase functions.

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Later on, Hörmander's local $L^2$ result was extended by R. Beals [3] and A. Greenleaf and G. Uhlmann [5] to the case of amplitudes in $S^0_{\frac{1}{2},\frac{1}{2}}$.

On the other hand, other classes of amplitudes and phase functions were studied. In [9] and [13], D. Robert and B. Helffer treated the symbol class $\Gamma^\mu_\rho$ and they considered phase functions satisfying certain properties. In [7, 12], Harrat-Senoussaoui and Messirdi-Senoussaoui treated the $L^2$-boundedness and $L^2$-compactness of $h$-FIO with symbol class just defined and $\varphi(x,\xi,y) = S(x,\xi) - \langle y,\xi\rangle$.

$$F_h u(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{i}{h}(S(x,\xi)-\langle y,\xi\rangle)} a(x,\xi)\, u(y)\, dy d\xi,$$

The aim of this work is to extend results obtained in [12, 7], the same hypothesis on the phase function are kept, and we mainly prove the continuity of $h$-FIO on $H^s(\mathbb{R}^n)$ when the weight of the amplitude $a$ is bounded. Using the estimate given in [1] for pseudodifferential operators, we also establish an estimate of $\|F_h\|_{\mathcal{L}(H^s(\mathbb{R}^n))}$.

Noted that in Hörmander's class this result is not true in general. In fact, in [8, 15] the author gave an example of FIO with symbol belonging to $\bigcap_{0<\rho<1} S^0_{\rho,1}$ that cannot be extended as a bounded operator on $L^2$ (same for $H^s$).

## 2. A general class of $h$-FIO.

**Definition 1.** Let $\Omega$ be an open set in $\mathbb{R}^n_x \times \mathbb{R}^N_\xi \times \mathbb{R}^n_y$, $\mu \in \mathbb{R}$ and $\rho \in [0,1]$. The space of *amplitudes* $\Gamma^\mu_\rho(\Omega)$ is the set of smooth functions $a\colon \Omega \to \mathbb{C}$ such that

$$\left|\partial^\alpha_x \partial^\beta_\xi \partial^\gamma_y a(x,\xi,y)\right| \le C_{\alpha\beta\gamma} \lambda^{\mu-\rho(|\alpha|+|\beta|+|\gamma|)}(x,\xi,y),$$

uniformly in $(x,\xi,y) \in \Omega$ for all $(\alpha,\beta,\gamma) \in \mathbb{N}^n \times \mathbb{N}^N \times \mathbb{N}^n$. Moreover, let

$$|a|^{\mu,\rho}_k := \max_{|\alpha|+|\beta|+|\gamma|\le k}\ \sup_{(x,\xi,y)\in\Omega} \lambda^{-\mu+\rho(|\alpha|+|\beta|+|\gamma|)}(x,\xi,y)\left|\partial^\alpha_x\partial^\beta_\xi\partial^\gamma_y a(x,\xi,y)\right|, k \in \mathbb{N}$$

be the associated sequence of semi-norms, where $\lambda(x,\xi,y) = \left(1+|x|^2+|y|^2+|\xi|^2\right)^{1/2}$ is called the weight.

When $\Omega = \mathbb{R}^n_x \times \mathbb{R}^N_\xi \times \mathbb{R}^n_y$, we denote $\Gamma^\mu_\rho(\Omega) = \Gamma^\mu_\rho$.

**Proposition 1.** *Let $\mu,\nu \in \mathbb{R}$ and $\rho,\delta \in [0,1]$*

1. *If $a \in \Gamma^\mu_\rho(\Omega)$ then $\partial^\alpha a \in \Gamma^{\mu-\rho|\alpha|}_\rho(\Omega)$;*
2. *If $a \in \Gamma^\mu_\rho(\Omega)$ and $b \in \Gamma^\nu_\rho(\Omega)$ then $ab \in \Gamma^{\mu+\nu}_\rho(\Omega)$;*
3. *If $\rho \le \delta$, then $\Gamma^\mu_\delta(\Omega) \subset \Gamma^\mu_\rho(\Omega)$.*

*Proof.* The proof is based on Leibniz's formula. □

Now, we consider the class of Fourier integral operators

$$(I(a,\varphi;h)u)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^N} e^{\frac{i}{h}\varphi(x,\xi,y)} a(x,\xi,y)\, u(y)\, dy d\xi, \tag{1}$$

with $u \in S(\mathbb{R}^n)$.

Let $a \in \Gamma^\mu_\rho$ and $\varphi$ be a phase function which satisfies the following conditions:

$(H_1)$ $\varphi : \mathbb{R}_x^n \times \mathbb{R}_\xi^N \times \mathbb{R}_y^n \to \mathbb{R}$ is a smooth function.

$(H_2)$ $\forall (\alpha, \beta, \gamma) \in \mathbb{N}^n \times \mathbb{N}^N \times \mathbb{N}^n, \exists C_{\alpha,\beta,\gamma} \geq 0;$

$$\left|\partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma \varphi (x, \xi, y)\right| \leq C_{\alpha,\beta,\gamma} \lambda^{(2-|\alpha|-|\beta|-|\gamma|)_+} (x, \xi, y),$$

where $(2 - |\alpha| - |\beta| - |\gamma|)_+ = \max (2 - |\alpha| - |\beta| - |\gamma|, 0)$.

$(H_3)$ There exist $K_1, K_2 > 0$ such that

$$K_1 \lambda (x, \xi, y) \leq \lambda (\partial_y \varphi, \partial_\xi \varphi, y) \leq K_2 \lambda (x, \xi, y), \forall (x, \xi, y) \in \mathbb{R}_x^n \times \mathbb{R}_\xi^N \times \mathbb{R}_y^n.$$

$(H_3^*)$ There exist $K_1^*, K_2^* > 0$ such that

$$K_1^* \lambda (x, \xi, y) \leq \lambda (x, \partial_\xi \varphi, \partial_x \varphi) \leq K_2^* \lambda (x, \xi, y), \forall (x, \xi, y) \in \mathbb{R}_x^n \times \mathbb{R}_\xi^N \times \mathbb{R}_y^n.$$

To give a meaning to the right-hand side of $(1)$, we use the oscillatory integral method. So we consider $g \in S\left(\mathbb{R}_x^n \times \mathbb{R}_\xi^N \times \mathbb{R}_y^n\right)$, $g(0) = 1$. If $a \in \Gamma_0^\mu$, we define

$$a_p (x, \xi, y) = g\left(\frac{x}{p}, \frac{\xi}{p}, \frac{y}{p}\right) a (x, \xi, y), \ p > 0.$$

Now we are able to state the following result.

**Theorem 1.** *If $\varphi$ satisfies $(H_1), (H_2), (H_3)$ and $(H_3^*)$, and if $a \in \Gamma_0^\mu$, then*

1. *For all $u \in S(\mathbb{R}^n)$, $\lim_{p\to\infty} [(I (a_p, \varphi; h) u)] (x)$ exists for every $x \in \mathbb{R}^n$ and is independent of the choice of the function $g$. We set then*

$$(I (a, \varphi; h) u) := \lim_{p\to\infty} (I (a_p, \varphi, h) u).$$

2. *$I (a, \varphi; h) \in \mathcal{L} (S (\mathbb{R}^n))$ and $I (a, \varphi; h) \in \mathcal{L} (S' (\mathbb{R}^n))$.*

The proof of the above theorem can be found in [9].

## 3. Sobolev spaces.

**Definition 2.** Let $s \in \mathbb{R}$. Then the *Bessel potential space* $H^s (\mathbb{R}^n)$ is defined as

$$H^s (\mathbb{R}^n) := \left\{u \in S' (\mathbb{R}^n) : \langle D_x\rangle^s u \in L^2 (\mathbb{R}^n)\right\}, \quad \|u\|_{H^s} := \|u\|_{2,s} := \|\langle D_x\rangle^s u\|_2,$$

where $\langle\xi\rangle := \lambda (\xi)$ and for all $u \in S' (\mathbb{R}^n)$ we have

$$\langle D_x\rangle^s u = \mathcal{F}^{-1} [\langle\xi\rangle^s \widehat{u}], \text{ with } \mathcal{F}^{-1}[u](x) := (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} g(\xi) d\xi.$$

**Remark 1.** By definition $\langle D_x\rangle^s$ is an isomorphism from $H^s (\mathbb{R}^n)$ to $L^2 (\mathbb{R}^n)$. Hence $H^s (\mathbb{R}^n)$ normed by $\|.\|_{2,s}$ is a Banach space. Moreover, $(u, v)_{H^s} := (\langle D_x\rangle^s u, \langle D_x\rangle^s v)_{L^2}$ is a scalar product on $H^s (\mathbb{R}^n)$ and $\|u\|_{2,s}^2 = (u, u)_{H^s}$. Thus $H^s (\mathbb{R}^n)$ is a Hilbert space.

**Proposition 2.** *If $x \to a (x)$ belongs to $\Gamma_k^m (\mathbb{R}_x^n)$, then $(x, y) \to a (x)$ belongs to $\Gamma_k^m \left(\mathbb{R}_x^n \times \mathbb{R}_y^N\right)$ for $k \in \{0, 1\}$.*

*Proof.* Denote $b(x,y) = a(x)$. We have to prove that $b(x,y) \in \Gamma_k^m\left(\mathbb{R}_x^n \times \mathbb{R}_y^N\right)$ with $k \in \{0,1\}$.

Let $(\alpha,\beta) \in \mathbb{N}^n \times \mathbb{N}^N$. If $|\beta| \geq 1$, then $\left|\partial_x^\alpha \partial_y^\beta b(x,y)\right| = 0$.

Otherwise, if $|\beta| = 0$,

$$|\partial_x^\alpha b(x,y)| = |\partial_x^\alpha a(x)| \leq C_\alpha \lambda^{m-|\alpha|}(x) \leq C_\alpha \lambda^{m-|\alpha|}(x,y).$$

So $b(x,y) \in \Gamma_k^m\left(\mathbb{R}_x^n \times \mathbb{R}_y^N\right)$. □

**Example 1.** We have that $\xi \mapsto \langle \xi \rangle^s$ belongs to $\Gamma_0^s(\mathbb{R}^n)$. Using Proposition 2, we deduce that $a(\xi,x) = \langle \xi \rangle^s \in \Gamma_0^s\left(\mathbb{R}_\xi^n \times \mathbb{R}_x^n\right)$.

**Lemma 1.** *Let $a \in \Gamma_0^m(\mathbb{R}^n \times \mathbb{R}^n)$. Then $Op_h(a) : H^{s+m}(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$ is bounded.*

*Proof.* Let $s,m \in \mathbb{R}$. Since $\langle D_x \rangle^{s+m} : H^{s+m}(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ and $\langle D_x \rangle^{-s} : L^2(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$ are linear isomorphisms, $Op_h(a) : H^{s+m}(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$ is a linear bounded operator if and only if

$$Op_h(b) := \langle D_x \rangle^s Op_h(a) \langle D_x \rangle^{-s-m} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

is a bounded operator.

We have the mapping

$$\Gamma_0^{m_1} \times \Gamma_0^{m_2} \ni (a_1,a_2) \mapsto a_1 \sharp a_2 \in \Gamma_0^{m_1+m_2},$$

where $a_1 \sharp a_2$ is the symbol of $Op_h(a_1)Op_h(a_2)$.

For $a_1(x,\xi) = \langle \xi \rangle^s$ and $a_2(x,\xi) = a(x,\xi)\langle \xi \rangle^{-s-m}$ one has $b \in \Gamma_0^0(\mathbb{R}^n \times \mathbb{R}^n)$.

By using Caldéron-Vaillancourt's theorem (see [4]), $Op_h(b)$ is bounded on $L^2(\mathbb{R}^n)$ □

**Corollary 1.** *If $a \in \Gamma_0^0(\mathbb{R}^n \times \mathbb{R}^n)$, then $Op(a) \in \mathcal{L}(H^s(\mathbb{R}^n))$. Moreover, there is some $k \in \mathbb{N}$ such that*

$$\|Op(a)\|_{\mathcal{L}(H^s(\mathbb{R}^n))} \leq C_s |a|_k^{0,0}. \tag{2}$$

**4. $H^s$-boundedness of $h$-FIO.** In this section we shall be interested in a particular case on the phase function $\varphi$.

$$\varphi(x,\xi,y) = S(x,\xi) - \langle y,\xi \rangle,$$

where $S \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R})$ satisfying

$(G_1)$ For all $(\alpha,\beta) \in \mathbb{N}^n \times \mathbb{N}^n$, there exist $C_{\alpha,\beta} > 0$, such that

$$\left|\partial_x^\alpha \partial_\xi^\beta S(x,\xi)\right| \leq C_{\alpha,\beta} \lambda(x,\xi)^{(2-|\alpha|-|\beta|)}.$$

$(G_2)$ There exists $\delta_0 > 0$ such that $\inf\limits_{x,\xi \in \mathbb{R}^n} \left|\det \dfrac{\partial^2 S}{\partial x \partial \xi}(x,\xi)\right| \geq \delta_0$.

**Lemma 2.** *If $S$ satisfies $(G_1)$ and $(G_2)$, then $\varphi$ satisfies $(H_1)$, $(H_2)$, $(H_3)$ and $(H_3^*)$.*

*Proof.* $(H_1)$ and $(H_2)$ are trivially satisfied.

The conditions $(G_1)$ and $(G_2)$ implies the existence of $C_1, C_2 > 0$ such that

$$\begin{cases} |x| \leq C_1 \lambda(\xi, \partial_\xi S), & \text{for all } (x,\xi) \in \mathbb{R}^{2n}, \\ |\xi| \leq C_2 \lambda(x, \partial_x S), & \text{for all } (x,\xi) \in \mathbb{R}^{2n}. \end{cases} \tag{3}$$

From (3) we have

$$\lambda(x,y,\xi) \leq \lambda(x,\xi) + \lambda(y) \leq C_3(\lambda(\xi,\partial_\xi S) + \lambda(y)), \ C_3 > 0.$$

Also, we have $\partial_{y_j}\varphi = -\xi_j$; and $\partial_{\xi_j}\varphi = \partial_{\xi_j}S - y_j$, so

$$\lambda(\xi, \partial_\xi S) = \lambda(\partial_y\varphi, \partial_\xi\varphi + y) \le 2\lambda(\partial_y\varphi, \partial_\xi\varphi, y),$$

which finally gives for some $C_4 > 0$,

$$\lambda(x,\xi,y) \le C_3\left(2\lambda(\partial_y\varphi, \partial_\xi\varphi, y)\right) \le \frac{1}{C_4}\lambda(\partial_y\varphi, \partial_\xi\varphi, y).$$

The second inequality in $(H_3)$ is a consequence of $(3)$.

By a similar argument we can show $(H_3^*)$. □

**Example 2.** Consider the function given by

$$S(x,\xi) = \sum_{|\alpha|+|\beta|=2} K_{\alpha,\beta}x^\alpha\xi^\beta,$$

where $K_{\alpha,\beta}$ are real constants.

For all $(x,\xi) \in \mathbb{R}^{2n}$, $S(x,\xi)$ satisfies $(G_1)$ and $(G_2)$.

We have the following result concerning the boundedness on $H^s(\mathbb{R}^n)$ of the Fourier integral operator defined by

$$(F_h u)(x) = (2\pi h)^{-n}\iint e^{\frac{i}{h}(S(x,\xi)-\langle y,\xi\rangle)}a(x,\xi)\,u(y)\,dyd\xi.$$

**Proposition 3.** *Let $F$ be the integral operator of the distribution kernel*

$$K(x,y) = \int_{\mathbb{R}^n} e^{\frac{i}{h}(S(x,\xi)-\langle y,\xi\rangle)}a(x,\xi)\,\widehat{d_h\xi}, \tag{4}$$

*where $\widehat{d_h\xi} = (2\pi h)^{-n}d\xi$, $a \in \Gamma_k^m\left(\mathbb{R}^{2n}_{x,\xi}\right)$, $k \in \{0,1\}$ and $S$ satisfies $(G_1)$ and $(G_2)$. Then $FF^*$ and $F^*F$ are pseudodifferential operators with symbol in $\Gamma_k^{2m}(\mathbb{R}^{2n})$, $k \in \{0,1\}$, given by*

$$\sigma(F_hF_h^*)(x,\partial_x S(x,\xi)) \equiv |a(x,\xi)|^2\left|\left(\det\frac{\partial^2 S}{\partial\xi\partial x}\right)^{-1}(x,\xi)\right|,$$

$$\sigma(F_h^*F_h)(\partial_\xi S(x,\xi),\xi) \equiv |a(x,\xi)|^2\left|\left(\det\frac{\partial^2 S}{\partial\xi\partial x}\right)^{-1}(x,\xi)\right|,$$

*we denote here $a \equiv b$ for $a,b \in \Gamma_k^{2p}(\mathbb{R}^{2n})$ if $(a-b) \in \Gamma_k^{2p-2}(\mathbb{R}^{2n})$ and $\sigma$ stands for the symbol.*

See [2, 12] for the proof of Proposition 3.

**Theorem 2.** *Let $F$ be the integral operator with the distribution kernel*

$$K(x,y) = \int_{\mathbb{R}^n} e^{\frac{i}{h}(S(x,\xi)-\langle y,\xi\rangle)}a(x,y)\,\widehat{d_h\xi},$$

*where $a \in \Gamma_0^m\left(\mathbb{R}^{2n}_{x,\xi}\right)$ and $S$ satisfies $(G_1)$ and $(G_2)$. Then for any $m$ such that $m \le 0$, $F$ can be extended as a bounded linear mapping on $H^s(\mathbb{R}^n)$.*

*Proof.* It follows from Proposition 3 that $F^*F$ is a pseudodifferential operator with symbol in $\Gamma_0^{2m}(\mathbb{R}^{2n})$.

(1) If $m \leq 0$, so we can apply inequality (2) in the corollary (1) for $F_h^* F_h$ and obtain the existence of a positive constant $C_s$ and $k \in \mathbb{N}$ such that

$$\|F_h^* F_h\|_{\mathcal{L}(H^s(\mathbb{R}^n))} \leq C_s \left|\sigma\left(F_h^* F_h\right)\right|_k^{0,0} < +\infty$$

Hence, for all $u \in S(\mathbb{R}^n)$,

$$\|F_h u\|_{H^s(\mathbb{R}^n)} \leq \|F_h^* F_h\|_{\mathcal{L}(H^s(\mathbb{R}^n))}^{\frac{1}{2}} \|u\|_{H^s(\mathbb{R}^n)} \leq \left(C_s \left|\sigma\left(F_h^* F_h\right)\right|_k^{0,0}\right)^{\frac{1}{2}} \|u\|_{H^s(\mathbb{R}^n)}.$$

Thus $F_h$ is a bounded linear operator on $H^s(\mathbb{R}^n)$. □

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