Entire Multivariate Vector-Valued Functions of Bounded $L$-Index: Analog of Fricke’s Theorem

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1. Introduction. A concept of bounded index for entire function ([14]) draws attention of many mathematician (see a full bibliography in [7, 17, 18, 11, 8]) to investigations of the corresponding function class and possible applications of this concept. It is interesting with its connections to the value distribution theory and analytic theory of differential equation ([11, 18, 4]). For example, every entire function has bounded value distribution if and only if its derivative has bounded index ([13]).

Recently, F. Nuray and R. Patterson ([16]) proposed a generalization of the concept of bounded index for entire bivariate functions from $\mathbb{C}^2$ into $\mathbb{C}^n$ by replacing the absolute value in the definition of an entire function of bounded index by the maximum of the absolute values of the components. If the components of a $\mathbb{C}^n$-valued bivariate entire function are of bounded index, then the function is also of bounded index. They presented sufficient

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conditions providing index boundedness of bivariate vector-valued entire solutions of certain system of partial differential equations with polynomial coefficients.

In recent papers [1, 2, 3] V. Baksa, A. Bandura, O. Skaskiv considered vector-valued functions having bounded L-index in joint variables which are analytic in the unit ball. They also extended previous investigations of analytic functions in the unit ball ([6]).

Our present goal is to give a completed form of investigations of F. Nuray and R. Patterson from [16]. In particular, they used some propositions without strict proofs for entire bivariate vector-valued functions. Moreover, there was considered a concept of bounded index for functions from \( \mathbb{C}^2 \) into \( \mathbb{C}^n \). Nevertheless, there is known a more general concept of bounded L-index in joint variables ([8]) with applications to system of partial differential equations ([11]).

Therefore, in our present investigation we will consider entire multivariate vector-valued functions from \( \mathbb{C}^n \) into \( \mathbb{C}^p \) and introduce concept of bounded L-index in joint variables for these functions.

2. Notations and definitions. Here we use some standard notations (see [1, 8]). Let \( \mathbb{R}_+ = [0; +\infty) \), \( 0 = (0, \ldots, 0) \in \mathbb{R}_+^n \), \( 1 = (1, \ldots, 1) \in \mathbb{R}_+^n \), \( e_j = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}_+^n \), \( R \subset \mathbb{R}_+^n \), \( R = (r_1, \ldots, r_n) \in \mathbb{R}_+^n \), \( |z| = \sqrt{|z_1|^2 + \ldots + |z_n|^2} \), \( z = (z_1, \ldots, z_n) \in \mathbb{C}^n \). For \( A = (a_1, \ldots, a_n) \in \mathbb{R}_+^n \), \( B = (b_1, \ldots, b_n) \in \mathbb{R}_+^n \), we will use formal notations without violation of the existence of these expressions: \( AB = (a_1b_1, \ldots, a_nb_n) \), \( A/B = (a_1/b_1, \ldots, a_nb_n) \), \( A^B = (a_1^b_1, \ldots, a_n^b_n) \), and the notation \( A < B \) means that \( a_j < b_j \), \( j \in \{1, \ldots, n\} \); the relation \( A \leq B \) is defined in the similar way. For \( K = (k_1, \ldots, k_n) \in \mathbb{Z}_+^n \) let us denote \( K! = k_1!\ldots k_n! \). Addition, multiplication by scalar and conjugation in \( \mathbb{C}^n \) are defined componentwise. For \( a = (a_1, \ldots, a_n) \in \mathbb{C}^n \), \( b = (b_1, \ldots, b_n) \in \mathbb{C}^n \) we define \( \langle a, b \rangle = a_1\overline{b_1} + \ldots + a_nb_n \), where \( \overline{b_j} \) is the complex conjugate of \( b_j \).

For \( z_0 = (z_{0,1}, \ldots, z_{0,n}) \in \mathbb{C}^n \) and \( R = (r_1, \ldots, r_n) \in \mathbb{R}_+^n \) we denote by \( D^n(z_0, R) = \{ z \in \mathbb{C}^n : |z_1 - z_{0,1}| < r_1, \ldots, |z_n - z_{0,n}| < r_n \} \) the polyydisc, by \( T^n(z_0, R) = \{ z \in \mathbb{C}^n : |z_1 - z_{0,1}| = r_1, \ldots, |z_n - z_{0,n}| = r_n \} \) its skeleton. The closed polyydisc \( \{ z \in \mathbb{C}^n : |z_1 - z_{0,1}| \leq r_1, \ldots, |z_n - z_{0,n}| \leq r_n \} \) is denoted by \( D^n[z_0, R] \), \( D^n = D^n(0; 1) \), \( D = \{ z \in \mathbb{C} : |z| < 1 \} \).

Let \( F(z) = (f_1(z), \ldots, f_p(z)) \) be an entire vector-valued function in \( \mathbb{C}^n \), i.e. \( f_j : \mathbb{C}^n \to \mathbb{C} \) is an entire function for every \( j, 1 \leq j \leq p \). Then at a point \( a \in \mathbb{C}^n \) the function \( F(z) \) has a vector-valued Taylor expansion

\[
F(z) = \sum_{k=0}^{+\infty} \sum_{\|m\|=k} C_m(z - a)^m,
\]

where

\[
C_m = \frac{1}{m!} F^{(m)}(a) := \frac{1}{m!} (f_1^{(m)}(a), \ldots, f_p^{(m)}(a)), \quad f_j^{(m)}(a) := \frac{\partial^{\|m\|} f_j(z)}{\partial z^{m}} \bigg|_{z=a}
\]

for \( m = (m_1, \ldots, m_n) \in \mathbb{Z}_+^n \), \( a \in \mathbb{C}^n \).

Let \( G \subset \mathbb{C}^n \) be some domain and \( \| \cdot \|_p \) be a norm in \( \mathbb{C}^p \). Let \( L(z) = (l_1(z), \ldots, l_n(z)) \), where \( l_j(z) : G \to \mathbb{R}_+ \) be a positive continuous function. An analytic vector-valued function \( F : G \to \mathbb{C}^p \) is said to be of bounded L-index (in joint variables) in the domain \( G \), if there exists \( n_0 \in \mathbb{Z}_+ \) such that

\[
\forall z \in G \forall J \in \mathbb{Z}_+^n : \frac{|F^{(j)}(z)|_p}{J!|L^j(z)|} \leq \max \left\{ \frac{|F^{(K)}(z)|_p}{K!|L^K(z)|} : K = \mathbb{Z}_+^n, \|K\| \leq n_0 \right\}.
\]
The least such integer $n_0$ is called the L-index in joint variables of the vector-valued function $F$ and is denoted by $N(F, L, G, C^p)$. For $G = C^p$ we denote $N(F, L) := N(F, L, C^p, C^p)$, the function $F$ is called an entire vector-valued function of bounded L-index $N(F, L)$. The concept of boundedness of L-index in joint variables were considered for other classes of analytic functions. They are differed in domains of analyticity: the unit ball ([6]), the polydisc ([9]), the Cartesian product of the unit disc and complex plane ([10]), $n$-dimensional complex space ([8, 11]), slice analyticity ([5]).

By $Q^n$ we denote the class of functions $L: C^n \rightarrow R^n_+$ such that for any $j \in \{1, 2, \ldots, n\}$
$$\forall R \in R^n_+: \ 0 < \lambda_{1,j}(R) \leq \lambda_{2,j}(R) < \infty,$$
where $\lambda_{1,j}(R) = \inf_{z_0 \in C^n} \{l_j(z)/l_j(z_0) : z \in D^n[z_0, R/L(z_0)]\}$, $\lambda_{2,j}(R)$ is defined analogously with replacement inf by sup sup. Remark that $(\forall R \in R^n_+) : \ \lambda_{1,j}(R) \leq 1 \leq \lambda_{2,j}(R)$ and $(\forall 1 \leq j \leq n)(\forall R_1, R_2 \in R^n_+) : R_1 < R_2 \implies \lambda_{2,j}(R_1) \leq \lambda_{2,j}(R_2), \ \lambda_{1,j}(R_1) \geq \lambda_{1,j}(R_2)$.

3. Local behavior of partial derivatives of entire vector-valued functions having bounded L-index in joint variables. The following theorem is basic in the theory of functions of bounded index. For various classes of analytic functions similar theorems are proved in [1, 10, 15, 17].

Theorem 1. Let $L \in Q^n$ and $|A|_p = \max\{|a_j| : 1 \leq j \leq p\}$ for $A = (a_1, \ldots, a_p) \in C^p$. An entire vector-valued function $F: C^n \rightarrow C^p$ has bounded L-index in joint variables if and only if for every $R \in R^n_+$ there exist $n_0 \in Z_+, p_0 > 0$ such that for all $z_0 \in C^n$ there exists $K_0 \in Z^n_+$, $\|K_0\| \leq n_0$, satisfying the inequality

$$\max\left\{\frac{|F^{(K)}(z)|_p}{K\|L^K(z)\|} : \|K\| \leq n_0, z \in D^n[z_0, R/L(z_0)]\right\} \leq p_0 \frac{|F^{(K_0)}(z_0)|_p}{K_0\|L^{K_0}(z_0)\|}. \quad (1)$$

Proof. Necessity. Let $F$ be an entire vector-valued function of bounded L-index in joint variables with $N = N(F, L) < \infty$. For any $R \in R^n_+$ we define

$$q = q(R) = \left[2(N + 1) \prod_{j=1}^n \left(\left(\lambda_{2,j}(R)\right)^{N+1} (\lambda_{1,j}(R))^{-N}\right)\right] + 1,$$
where $[x]$ stands for the entire part of a real number $x$. For $p_0 \in \{0, \ldots, q\}$ and $z_0 \in C^n$ we denote:

$$S_{p_0}(z_0, R) = \max\left\{\frac{|F^{(K)}(z)|_p}{K\|L^K(z)\|} : \|K\| \leq N, z \in D^n[z_0, p_0 R/(qL(z_0))]\right\},$$

$$S^*_p(z_0, R) = \max\left\{\frac{|F^{(K)}(z)|_p}{K\|L^K(z_0)\|} : \|K\| \leq N, z \in D^n[z_0, p_0 R/(qL(z_0))]\right\}.$$ We note that $D^n[z_0, p_0 R/(qL(z_0))] \subset D^n[z_0, R/L(z_0)]$, thus for all $z \in D^n[z_0, p_0 R/(qL(z_0))]$ by the definition of $\lambda_{1,j}(R)$ we have

$$L^K(z_0) = l_1^{k_1}(z_0) \cdots l_n^{k_n}(z_0) \leq \lambda_{1,1}^{-k_1}(R) \cdots \lambda_{1,n}^{-k_n}(R) = \lambda_1^{-K}(R), \quad K = (k_1, \ldots, k_n),$$
where $\lambda_1(R) := (\lambda_{1,1}(R), \ldots, \lambda_{1,n}(R)) \in R^n_+$. Hence,

$$S_{p_0}(z_0, R) = \max\left\{\frac{|F^{(K)}(z)|_p}{K\|L^K(z)\|} : \|K\| \leq N, z \in D^n[z_0, p_0 R/(qL(z_0))]\right\} =$$
\[ \begin{align*}
&= \max \left\{ \frac{|F'(K)(z)|}{K! L^K(z_0)} \cdot \frac{L^K(z_0)}{L^K(z)} : \|K\| \leq N, z \in \mathbb{D}^n[z_0, p_0 R/(q L(z_0))] \right\} \\
&\leq S_{p_0}^*(z_0, R) \max \{\lambda_1(R)^{-K} : \|K\| \leq N\} \leq S_{p_0}^*(z_0, R) \prod_{j=1}^n (\lambda_{1,j}(R))^{-N}. \quad (2)
\end{align*} \]

For all \( z \in \mathbb{D}^n[z_0, p_0 R/(q L(z_0))] \) by the definition of \( \lambda_{2,j}(R) \), for \( K = (k_1, \ldots, k_n) \) we have

\[ \begin{align*}
\frac{L^K(z)}{L^K(z_0)} &= \frac{l_1^{k_1}(z)}{l_1^{k_1}(z_0)} \cdots \frac{l_n^{k_n}(z)}{l_n^{k_n}(z_0)} \leq \lambda_{2,1}^{k_1}(R) \cdots \lambda_{2,n}^{k_n}(R) = \lambda_2^K(R),
\end{align*} \]

where \( \lambda_2(R) := (\lambda_{2,1}(R), \ldots, \lambda_{2,n}(R)) \in \mathbb{R}^n_+ \). Hence, one has:

\[ \begin{align*}
S_{p_0}^*(z_0, R) &\leq \max \left\{ \frac{|F'(K)(z)|}{K! L^K(z)} \lambda_2(R)^K : \|K\| \leq N, z \in \mathbb{D}^n[z_0, p_0 R/(q L(z_0))] \right\} \leq \\
&\leq S_{p_0}^*(z_0, R) \prod_{j=1}^n (\lambda_{2,j}(R))^N. \quad (4)
\end{align*} \]

Let \( K_{p_0} \in \mathbb{Z}_+^n, \|K_{p_0}\| \leq N \) and \( z_s \in \mathbb{D}^n[z_0, p_0 R/(q L(z_0))] \) be such that

\[ S_{p_0}^*(z_0, R) = \frac{|F'(K_{p_0})(z_s)|}{K_{p_0}! L^{K_{p_0}}(z_0)}. \quad (5) \]

Since by the maximum modulus principle \( z_s \in \mathbb{T}^n(z_0, p_0 R/(q L(z_0))) \), therefore \( z_s \neq z_0 \).

We choose \( \tilde{z} = z_0 + \frac{p_0 - 1}{p_0} (z_s - z_0) \). Then for \( \tilde{z} = (\tilde{z}^{(1)}, \ldots, \tilde{z}^{(n)}) \), \( z_0 = (z_0^{(1)}, \ldots, z_0^{(n)}) \), \( z_s = (z_s^{(1)}, \ldots, z_s^{(n)}) \), \( 1 \leq j \leq n \) sequentially we have

\[ |\tilde{z}^{(j)} - z_0^{(j)}| = \frac{p_0 - 1}{p_0} |z_s^{(j)} - z_0^{(j)}| = \frac{p_0 - 1}{p_0} \frac{p_0 r_j}{q l_j(z_0)}, \quad (6) \]

\[ |\tilde{z}^{(j)} - z_s^{(j)}| = |z_0^{(j)} + \frac{p_0 - 1}{p_0} (z_s^{(j)} - z_0^{(j)}) - z_s^{(j)}| = \frac{1}{p_0} |z_0^{(j)} - z_s^{(j)}| = \frac{r_j}{q l_j(z_0)} \]. \quad (7)

We obtain \( \tilde{z} \in \mathbb{D}^n[z_0, (p_0 - 1)R/(q(R)L(z_0))] \) and thus \( S_{p_0-1}^*(z_0, R) \geq \frac{|F'(K_{p_0})(\tilde{z})|}{K_{p_0}! L^{K_{p_0}}(z_0)}. \)

Remark that

\[ \frac{d}{dt} \|F'(K_{p_0})(\tilde{z} + t(z_s - \tilde{z}))\| \leq \sum_{j=1}^n \left( |z_s^{(j)} - \tilde{z}^{(j)}| \cdot \|F'(K_{p_0} + \epsilon_j)(\tilde{z} + t(z_s - \tilde{z}))\| \right) \]

Then, from (5) by the mean value theorem we have

\[ \begin{align*}
0 &\leq S_{p_0}^*(z_0, R) - S_{p_{0-1}}^*(z_0, R) \leq \\
&\leq \frac{|F'(K_{p_0})(z_s)|}{K_{p_0}! L^{K_{p_0}}(z_0)} \cdot \frac{|F'(K_{p_0})(\tilde{z})|}{p} = \frac{1}{K_{p_0}! L^{K_{p_0}}(z_0)} \int_0^1 \frac{d}{dt} |F'(K_{p_0})(\tilde{z} + t(z_s - \tilde{z}))|p dt \leq \\
&\leq \frac{1}{K_{p_0}! L^{K_{p_0}}(z_0)} \int_0^1 \sum_{j=1}^n \left( |z_s^{(j)} - \tilde{z}^{(j)}| \cdot |F'(K_{p_0} + \epsilon_j)(\tilde{z} + t(z_s - \tilde{z}))|p \right) dt = 
\end{align*} \]
\[ \frac{1}{K_{p_0}^n L^{K_{p_0}}(z_0)} \sum_{j=1}^{n} \left( |z_j - \bar{z}_j| \cdot |F(K_{p_0}^n + e_j)(\bar{z} + t^*(z_0 - \bar{z}))| \right)_p, \]  \tag{8}

where \( 0 \leq t^* \leq 1 \), and \((\bar{z} + t^*(z_0 - \bar{z})) \in \mathbb{D}^n [z_0, p_0 R/(q L(z_0))]. \)

For \( z \in \mathbb{D}^n [z_0, p_0 R/(q L(z_0))] \) and \( J = (j_1, \ldots, j_n) \in \mathbb{Z}^+_n : \|J\| \leq N + 1 \), by the definition of the numbers \( N = N(F, L) \) and \( \lambda_{2,j}(p_0 R/q) \), we have

\[
\frac{|F(J)(z)|_p}{J! L^J(z_0)} = \frac{|F(J)(z)|_p}{J! L^J(z_0)} \cdot \frac{L^J(z)}{L^J(z_0)} \leq \frac{|F(J)(z)|_p}{J! L^J(z_0)} \max \left\{ \frac{L^J(z)}{L^J(z_0)} : \|J\| \leq N + 1 \right\} \leq \max \left\{ \frac{|F(K)(z)|_p}{K! L^K(z)} : \|K\| \leq N \right\} \cdot \prod_{j=1}^{n} \left( \lambda_{2,j}(p_0 R/q) \right)^{N+1} \leq \prod_{j=1}^{n} \left( \lambda_{2,j}(R) \right)^{N+1} \cdot \max \left\{ \frac{|F(K)(z)|_p}{K! L^K(z)} : \|K\| \leq N \right\} = \prod_{j=1}^{n} \left( \lambda_{2,j}(R) \right)^{N+1} \cdot S_{p_0}(z_0, R) \leq \prod_{j=1}^{n} \left( \lambda_{2,j}(R) \right)^{N+1} \cdot S_{p_0}^*(z_0, R) \cdot \prod_{j=1}^{n} \left( \lambda_{1,j}(R) \right)^{-N}.
\]

Hence, and from (8), (6) we obtain

\[
0 \leq S_{p_0}^*(z_0, R) - S_{p_0-1}^*(z_0, R) \leq \sum_{j=1}^{n} \left( \left| z_j - \bar{z}_j \right| \cdot \frac{(K_{p_0} + e_j)L^{K_{p_0} + e_j}(z_0)}{K_{p_0}^n L^{K_{p_0}}(z_0)} \cdot \|F(K_{p_0}^n + e_j)(\bar{z} + t^*(z_0 - \bar{z}))\| \right) \leq \prod_{j=1}^{n} \left( \lambda_{2,j}(R) \right)^{N+1} \left( \lambda_{1,j}(R) \right)^{-N} S_{p_0}^*(z_0, R) \times \sum_{j=1}^{n} \left| z_j - \bar{z}_j \right| \langle e_j, K_{p_0} + e_j \rangle L^{e_j}(z_0).
\]

From (7) we have \( \sum_{j=1}^{n} \left| z_j - \bar{z}_j \right| \langle e_j, K_{p_0} + e_j \rangle L^{e_j}(z_0) = \frac{1}{q(R)} \sum_{j=1}^{n} \langle e_j, K_{p_0} + e_j \rangle R^{e_j} \leq (N + 1) \sum_{j=1}^{n} R^{e_j} = (N + 1)\|R\|. \) Therefore, using the of choice of \( q(R) \) we get

\[
S_{p_0}^*(z_0, R) - S_{p_0-1}^*(z_0, R) \leq \prod_{j=1}^{n} \left( \left( \lambda_{2,j}(R) \right)^{N+1} \left( \lambda_{1,j}(R) \right)^{-N} \right) S_{p_0}^*(z_0, R) \frac{R}{q(R)} (N+1)\|R\| \leq S_{p_0}^*(z_0, R) \frac{2}{2}.
\]

It follows that \( S_{p_0}^*(z_0, R) \leq 2S_{p_0-1}^*(z_0, R) \) and in view of (2) and (4) one has

\[
S_{p_0}(z_0, R) \leq 2 \prod_{j=1}^{n} \left( \left( \lambda_{1,j}(R) \right)^{-N} \right) S_{p_0}^*(z_0, R) \leq 2 \prod_{j=1}^{n} \left( \left( \lambda_{2,j}(R) \right)^{-N} \right) S_{p_0-1}(z_0, R).
\]

Then we consequently obtain

\[
S_q(z_0, R) \leq 2^q \prod_{j=1}^{n} \left( \left( \lambda_{1,j}(R) \right)^{-N} \right)^q S_{p_0}(z_0, R),
\]

\[
\max \left\{ \frac{|F(K)(z)|_p}{K! L^K(z)} : \|K\| \leq N, z \in \mathbb{D}^p [z_0, R/L(z_0)] \right\} =
\]
We note that
\[ K \subseteq \mathbb{D}^p [z_0, qR/(qL(z_0))] \]
and for some
\[ \lambda_{ij}(R) \]
and for some
\[ z_0 \in \mathbb{C}^n \]
and some
\[ K_0 \] such that
\[ ||K_0|| \leq N. \] The necessity of condition (1) is proved.

**Sufficiency.** Assume that for every
\[ R \in \mathbb{R}^n_+ \]
there exist
\[ n_0 \in \mathbb{Z}_+ \]
and for some
\[ K_0 \in \mathbb{Z}^n_+ \]
and for some
\[ ||K_0|| \leq n_0 \], inequality (1) holds. By Cauchy’s integral formula we have
\[ \forall z_0 \in \mathbb{C}^n \], \( \forall K \in \mathbb{Z}^n_+ \), \( \forall S \in \mathbb{Z}^n_+ \):
\[ \int_{\mathbb{T}(z_0, R/L(z_0))} \frac{F(K)(z)}{(z - z_0)^n} dz = \frac{F(K)(z)}{S!} \]
Therefore,
\[ |F(K+S)(z_0)|_p \leq \frac{p_0K!}{K_0!(2\pi)^n} |F(K_0)(z_0)|_p \frac{L^{S+1}(z_0)}{L^{K_0}(z_0) R^{S+1}} \int_{\mathbb{T}(z_0, R/L(z_0))} L^K(z) dz |dz| \]
But, for all
\[ z \in \mathbb{D}^n [z_0, R/L(z_0)] \] by the definition of
\[ \lambda_{ij}(R) \]
we have
\[ L^K(z) = L^K(z_0) \cdot \frac{L^K(z)}{L^K(z_0)} = L^K(z_0) \cdot \frac{z_1z_2 \ldots z_n}{z_1z_2 \ldots z_n} \]
\[ \leq L^K(z_0) \lambda_{2,1}(R) \ldots \lambda_{2,n}(R) = L^K(z_0) \lambda_{2}^K(R), \quad K = (k_1, \ldots, k_p). \]
Hence,
\[ \frac{|F(K+S)(z_0)|_p}{(K+S)! L^{K+S}(z_0)} \leq \frac{|F(K_0)(z_0)|_p p_0K!S! \lambda_{2}^K(R)}{(K_0)! L^{K_0}(z_0) (K+S)! R^{S}}. \]
We note that
\[ \frac{K^{(S)}_{(K+S)}}{K^{(S)}_{(K+S)}} \leq 1 \quad (\forall K, S \in \mathbb{Z}^n_+), \quad R^S \rightarrow +\infty \quad \text{as} \quad ||S|| \rightarrow +\infty \quad \text{for every} \quad R \in (1, +\infty)^n. \]
Therefore, for each fixed
\[ R \in (1, +\infty)^n \]
and every
\[ K \in \mathbb{Z}^n_+ \], \( ||K|| \leq n_0 \), there exists
\[ s_0 \in \mathbb{N} \] such that for every
\[ S \in \mathbb{Z}^n_+ \], \( ||S|| \geq s_0 \), the inequality
\[ \frac{p_0K!S! \lambda_{2}^K(R)}{(K+S)! R^{S}} \leq 1 \]
holds. Then, in view of (10), one has
\[ \frac{|F(K+S)(z_0)|_p}{(K+S)! L^{K+S}(z_0)} \leq \frac{|F(K_0)(z_0)|_p}{K_0! L^{K_0}(z_0)} \]
for all
\[ K, S \] such that
\[ ||K_0|| \leq n_0, ||S|| \geq s_0 \]. It implies that
\[ \forall z \in \mathbb{C}^n \quad \forall I \in \mathbb{Z}^n_+ : \]
\[ \frac{|F^I(z)|_p}{I! L^I(z)} \leq \max \left\{ \frac{|F^K(z)|_p}{K! L^K(z)} : K \in \mathbb{Z}^p_+, ||K|| \leq s_0 + n_0 \right\} \]
where
\[ s_0 \] and
\[ n_0 \] do not depend on
\[ z_0 \]. Then the entire vector-valued function
\[ F \] has bounded
\[ L \text{-index in joint variables} \quad N(F, L) \leq s_0 + n_0. \] The proof of theorem is complete.
Theorem 1 implies the following corollary.

**Corollary 1.** Let $L \in \mathbb{Q}^p$ and $\| \cdot \|_0$ be some norm in $\mathbb{C}^p$. An entire vector-function $F : \mathbb{C}^p \to \mathbb{C}^p$ has bounded $L$-index in joint variables in sup-norm if and only if it has bounded $L$-index in joint variables in the norm $\| \cdot \|_0$.

**Proof.** Recall that ([12]) if $\| \cdot \|_1$ and $\| \cdot \|_2$ are two norms in $\mathbb{C}^p$, then there exist constants $C_1, C_2 \in (0, +\infty)$ such that $C_1 \|w\|_1 \leq \|w\|_2 \leq C_2 \|w\|_1$ for every $w \in \mathbb{C}^p$. Thus, for all $K \in \mathbb{Z}^p_+$ and for all $z \in \mathbb{C}^p$ we obtain

$$C_1 \|F^{(K)}(z)\| \leq \|F^{(K)}(z)\|_0 \leq C_2 \|F^{(K)}(z)\|,$$

where $\| \cdot \|$ is the sup-norm. Using the given inequalities and repeating arguments from Theorem 1 for the case of the Euclidean norm we can verify the equivalence of these norms for vector-functions having bounded $L$-index in joint variables.

From Corollary 1, in particular, it follows that instead of the sup-norm $\|A\| = \max_{1 \leq j \leq p} |a_j|$ one can consider in Theorem 1 the Euclidean norm $\|A\|_E = \sqrt{|a_1|^2 + \ldots + |a_p|^2}$, where $A = (a_1, \ldots, a_p) \in \mathbb{C}^p$.

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