УДК 517.555

V. P. Baksa, A. I. Bandura

ENTIRE MULTIVARIATE VECTOR-VALUED FUNCTIONS OF BOUNDED L-INDEX: ANALOG OF FRICKE'S THEOREM

V. P. Baksa, A. I. Bandura. *Entire multivariate vector-valued functions of bounded* **L**-*index: analog of Fricke's theorem*, Mat. Stud. **54** (2020), 56–63.

We consider a class of vector-valued entire functions $F\colon \mathbb{C}^n \to \mathbb{C}^p$. For this class of functions there is introduced a concept of boundedness of **L**-index in joint variables.

Let $|\cdot|_p$ be a norm in $\mathbb{C}^p$. Let $\mathbf{L}(z) = (l_1(z), \dots, l_n(z))$, where $l_j(z)\colon \mathbb{C}^n \to \mathbb{R}_+$ is a positive continuous function. An entire vector-valued function $F\colon \mathbb{C}^n \to \mathbb{C}^p$ is said to be of bounded **L**-index (in joint variables), if there exists $n_0 \in \mathbb{Z}_+$ such that

$$\forall z \in G \ \ \forall J \in \mathbb{Z}^n_+\colon \quad \frac{|F^{(J)}(z)|_p}{J!\mathbf{L}^J(z)} \le \max\left\{\frac{|F^{(K)}(z)|_p}{K!\mathbf{L}^K(z)}\colon K \in \mathbb{Z}^n_+, \|K\| \le n_0\right\}.$$

We assume the function $\mathbf{L}\colon \mathbb{C}^n \to \mathbb{R}^p_+$ such that $0 < \lambda_{1,j}(R) \le \lambda_{2,j}(R) < \infty$ for any $j \in \{1, 2, \dots, p\}$ and $\forall R \in \mathbb{R}^p_+$, where $\lambda_{1,j}(R) = \inf\limits_{z_0 \in \mathbb{C}^p} \inf\left\{l_j(z)/l_j(z_0)\colon z \in \mathbb{D}^n[z_0, R/\mathbf{L}(z_0)]\right\}$, $\lambda_{2,j}(R)$ is defined analogously with replacement inf by sup. It is proved the following theorem: *Let $|A|_p = \max\{|a_j|\colon 1 \le j \le p\}$ for $A = (a_1, \dots, a_p) \in \mathbb{C}^p$. An entire vector-valued function F has bounded* **L**-*index in joint variables if and only if for every $R \in \mathbb{R}^n_+$ there exist $n_0 \in \mathbb{Z}_+$, $p_0 > 0$ such that for all $z_0 \in \mathbb{C}^n$ there exists $K_0 \in \mathbb{Z}^n_+$, $\|K_0\| \le n_0$, satisfying inequality*

$$\max\left\{\frac{|F^{(K)}(z)|_p}{K!\mathbf{L}^K(z)}\colon \|K\| \le n_0, z \in \mathbb{D}^n[z_0, R/\mathbf{L}(z_0)]\right\} \le p_0 \frac{|F^{(K_0)}(z_0)|_p}{K_0!\mathbf{L}^{K_0}(z_0)},$$

where $\mathbb{D}^n[z_0, R] = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n\colon |z_1 - z_{0,1}| < r_1, \dots, |z_n - z_{0,n}| < r_n\}$ is the polydisc with $z_0 = (z_{0,1}, \dots, z_{0,n})$, $R = (r_1, \dots, r_n)$. This theorem is an analog of Fricke's Theorem obtained for entire functions of bounded index of one complex variable.

1. Introduction. A concept of bounded index for entire function ([14]) draws attention of many mathematician (see a full bibliography in [7, 17, 18, 11, 8]) to investigations of the corresponding function class and possible applications of this concept. It is interesting with its connections to the value distribution theory and analytic theory of differential equation ([11, 18, 4]). For example, every entire function has bounded value distribution if and only if its derivative has bounded index ([13]).

Recently, F. Nuray and R. Patterson ([16]) proposed a generalization of the concept of bounded index for entire bivariate functions from $\mathbb{C}^2$ into $\mathbb{C}^n$ by replacing the absolute value in the definition of an entire function of bounded index by the maximum of the absolute values of the components. If the components of a $\mathbb{C}^n$-valued bivariate entire function are of bounded index, then the function is also of bounded index. They presented sufficient

2010 *Mathematics Subject Classification:* 32A10, 32A17, 32A37.
Keywords: bounded index; bounded **L**-index in joint variables; entire function; maximum modulus; sup-norm; vector-valued function.
doi:10.30970/ms.54.1.56-63

conditions providing index boundedness of bivariate vector-valued entire solutions of certain system of partial differential equations with polynomial coefficients.

In recent papers [1, 2, 3] V. Baksa, A. Bandura, O. Skaskiv considered vector-valued functions having bounded **L**-index in joint variables which are analytic in the unit ball. They also extended previous investigations of analytic functions in the unit ball ([6]).

Our present goal is to give a completed form of investigations of F. Nuray and R. Patterson from [16]. In particular, they used some propositions without strict proofs for entire bivariate vector-valued functions. Moreover, there was considered a concept of bounded index for functions from $\mathbb{C}^2$ into $\mathbb{C}^n$. Nevertheless, there is known a more general concept of bounded **L**-index in joint variables ([8]) with applications to system of partial differential equations ([11]).

Therefore, in our present investigation we will consider entire multivariate vector-valued functions from $\mathbb{C}^n$ into $\mathbb{C}^p$ and introduce concept of bounded **L**-index in joint variables for these functions.

2. Notations and definitions. Here we use some standard notations (see [1, 8]). Let $\mathbb{R}_+=[0;+\infty)$, $\mathbf{0}=(0,\ldots,0)\in\mathbb{R}^n_+$, $\mathbf{1}=(1,\ldots,1)\in\mathbb{R}^n_+$, $\mathbf{e}_j=(0,\ldots,0,\underbrace{1}_{j\text{-th place}},0,\ldots,0)\in\mathbb{R}^n_+$, $R=(r_1,\ldots,r_n)\in\mathbb{R}^n_+$, $|z|=\sqrt{|z_1|^2+\ldots+|z_n|^2}$, $z=(z_1,\ldots,z_n)\in\mathbb{C}^n$. For $A=(a_1,\ldots,a_n)\in\mathbb{R}^n$, $B=(b_1,\ldots,b_n)\in\mathbb{R}^n$, we will use formal notations without violation of the existence of these expressions: $AB=(a_1b_1,\ldots,a_nb_n)$, $A/B=(a_1/b_1,\ldots,a_n/b_n)$, $A^B=(a_1^{b_1},\ldots,a_n^{b_n})$, and the notation $A<B$ means that $a_j<b_j$, $j\in\{1,\ldots,n\}$; the relation $A\le B$ is defined in the similar way. For $K=(k_1,\ldots,k_n)\in\mathbb{Z}^n_+$ let us denote $K!=k_1!\cdot\ldots\cdot k_n!$. Addition, multiplication by scalar and conjugation in $\mathbb{C}^n$ are defined componentwise. For $a=(a_1,\ldots,a_n)\in\mathbb{C}^n$, $b=(b_1,\ldots,b_n)\in\mathbb{C}^n$ we define $\langle a,b\rangle=a_1\overline{b}_1+\ldots+a_n\overline{b}_n$, where $\overline{b}_j$ is the complex conjugate of b_j.

For $z_0=(z_{0,1},\ldots,z_{0,n})\in\mathbb{C}^n$ and $R=(r_1,\ldots,r_n)\in\mathbb{R}^n_+$ we denote by $\mathbb{D}^n(z_0,R)=\{z\in\mathbb{C}^n\colon |z_1-z_{0,1}|<r_1,\ldots,|z_n-z_{0,n}|<r_n\}$ the polydisc, by $\mathbb{T}^n(z_0,R)=\{z\in\mathbb{C}^n\colon |z_1-z_{0,1}|=r_1,\ldots,|z_n-z_{0,n}|=r_n\}$ its skeleton. The closed polydisc $\{z\in\mathbb{C}^n\colon |z_1-z_{0,1}|\le r_1,\ldots,|z_n-z_{0,n}|\le r_n\}$ is denoted by $\mathbb{D}^n[z_0,R]$, $\mathbb{D}^n=\mathbb{D}^n(\mathbf{0};\mathbf{1})$, $\mathbb{D}=\{z\in\mathbb{C}\colon |z|<1\}$.

Let $F(z)=(f_1(z),\ldots,f_p(z))$ be an entire vector-valued function in $\mathbb{C}^n$, i.e. $f_j\colon\mathbb{C}^n\to\mathbb{C}$ is an entire function for every j, $1\le j\le p$. Then at a point $a\in\mathbb{C}^n$ the function $F(z)$ has a vector-valued Taylor expansion

$$F(z)=\sum_{k=0}^{+\infty}\sum_{\|m\|=k}C_m(z-a)^m,$$

where

$$C_m=\frac{1}{m!}F^{(m)}(a):=\frac{1}{m!}\Big(f_1^{(m)}(a),\ldots,f_p^{(m)}(a)\Big),\ f_j^{(m)}(a):=\frac{\partial^{\|m\|}f_j(z)}{\partial z^m}=\frac{\partial^{\|m\|}f_j(z)}{\partial z_1^{m_1}\cdot\ldots\cdot\partial z_n^{m_n}}\Big|_{z=a}$$

for $m=(m_1,\ldots,m_n)\in\mathbb{Z}^n_+$, $a\in\mathbb{C}^n$.

Let $G\subset\mathbb{C}^n$ be some domain and $|\cdot|_p$ be a norm in $\mathbb{C}^p$. Let $\mathbf{L}(z)=(l_1(z),\ldots,l_n(z))$, where $l_j(z)\colon G\to\mathbb{R}_+$ be a positive continuous function. An analytic vector-valued function $F\colon G\to\mathbb{C}^p$ is said to be *of bounded* **L***-index (in joint variables) in the domain* G, if there exists $n_0\in\mathbb{Z}_+$ such that

$$\forall z\in G\ \ \forall J\in\mathbb{Z}^n_+\colon\quad \frac{|F^{(J)}(z)|_p}{J!\mathbf{L}^J(z)}\le\max\left\{\frac{|F^{(K)}(z)|_p}{K!\mathbf{L}^K(z)}\colon K\in\mathbb{Z}^n_+,\|K\|\le n_0\right\}.$$

The least such integer n_0 is called the **L**-index in joint variables of the vector-valued function F and is denoted by $N(F, \mathbf{L}, G, \mathbb{C}^p)$. For $G = \mathbb{C}^p$ we denote $N(F, \mathbf{L}) := N(F, \mathbf{L}, \mathbb{C}^p, \mathbb{C}^p)$, the function F is called an *entire vector-valued function of bounded* **L**-*index* $N(F, \mathbf{L})$. The concept of boundedness of **L**-index in joint variables were considered for other classes of analytic functions. They are differed in domains of analyticity: the unit ball ([6]), the polydisc ([9]), the Cartesian product of the unit disc and complex plane ([10]), n-dimensional complex space ([8, 11]), slice analyticity ([5]).

By $\mathbb{Q}^n$ we denote the class of functions $\mathbf{L}\colon \mathbb{C}^n \to \mathbb{R}^n_+$ such that for any $j \in \{1, 2, \dots, n\}$

$$\forall R \in \mathbb{R}^n_+\colon\ 0 < \lambda_{1,j}(R) \le \lambda_{2,j}(R) < \infty,$$

where $\lambda_{1,j}(R) = \inf\limits_{z_0 \in \mathbb{C}^n} \inf \{l_j(z)/l_j(z_0)\colon z \in \mathbb{D}^n[z_0, R/\mathbf{L}(z_0)]\}$, $\lambda_{2,j}(R)$ is defined analogously with replacement inf inf by sup sup. Remark that $(\forall R \in \mathbb{R}^n_+)\colon\ \lambda_{1,j}(R) \le 1 \le \lambda_{2,j}(R)$ and

$$(\forall j, 1 \le j \le n)(\forall R_1, R_2 \in \mathbb{R}^n_+)\colon\ R_1 < R_2 \implies \lambda_{2,j}(R_1) \le \lambda_{2,j}(R_2),\ \lambda_{1,j}(R_1) \ge \lambda_{1,j}(R_2).$$

3. Local behavior of partial derivatives of entire vector-valued functions having bounded L-index in joint variables. The following theorem is basic in the theory of functions of bounded index. For various classes of analytic functions similar theorems are proved in [1, 10, 15, 17].

Theorem 1. *Let* $\mathbf{L} \in \mathbb{Q}^n$ *and* $|A|_p = \max\{|a_j|\colon 1 \le j \le p\}$ *for* $A = (a_1, \dots, a_p) \in \mathbb{C}^p$. *An entire vector-valued function* $F\colon \mathbb{C}^n \to \mathbb{C}^p$ *has bounded* **L**-*index in joint variables if and only if for every* $R \in \mathbb{R}^n_+$ *there exist* $n_0 \in \mathbb{Z}_+$, $p_0 > 0$ *such that for all* $z_0 \in \mathbb{C}^n$ *there exists* $K_0 \in \mathbb{Z}^n_+$, $\|K_0\| \le n_0$, *satisfying the inequality*

$$\max\left\{\frac{|F^{(K)}(z)|_p}{K!\mathbf{L}^K(z)}\colon \|K\| \le n_0, z \in \mathbb{D}^n[z_0, R/\mathbf{L}(z_0)]\right\} \le p_0 \frac{|F^{(K_0)}(z_0)|_p}{K_0!\mathbf{L}^{K_0}(z_0)}. \tag{1}$$

Proof. Necessity. Let F be an entire vector-valued function of bounded **L**-index in joint variables with $N = N(F, \mathbf{L}) < \infty$. For any $R \in \mathbb{R}^n_+$ we define

$$q = q(R) = \left[2(N+1) \prod_{j=1}^{n} \left(\left(\lambda_{2,j}(R)\right)^{N+1} \left(\lambda_{1,j}(R)\right)^{-N}\right) \|R\|\right] + 1,$$

where $[x]$ stands for the entire part of a real number x. For $p_0 \in \{0, ..., q\}$ and $z_0 \in \mathbb{C}^n$ we denote:

$$S_{p_0}(z_0, R) = \max\left\{\frac{|F^{(K)}(z)|_p}{K!\mathbf{L}^K(z)}\colon \|K\| \le N, z \in \mathbb{D}^n\left[z_0, p_0 R/(q\mathbf{L}(z_0))\right]\right\},$$

$$S^*_{p_0}(z_0, R) = \max\left\{\frac{|F^{(K)}(z)|_p}{K!\mathbf{L}^K(z_0)}\colon \|K\| \le N, z \in \mathbb{D}^n\left[z_0, p_0 R/(q\mathbf{L}(z_0))\right]\right\}.$$

We note that $\mathbb{D}^n[z_0, p_0 R/(q\mathbf{L}(z_0))] \subset \mathbb{D}^n[z_0, R/\mathbf{L}(z_0)]$, thus for all $z \in \mathbb{D}^n\left[z_0, p_0 R/(q\mathbf{L}(z_0))\right]$ by the definition of $\lambda_{1,j}(R)$ we have

$$\frac{\mathbf{L}^K(z_0)}{\mathbf{L}^K(z)} = \frac{{l_1}^{k_1}(z_0)}{{l_1}^{k_1}(z)} \cdot \ldots \cdot \frac{{l_n}^{k_p}(z_0)}{{l_n}^{k_p}(z)} \le \lambda_{1,1}^{-k_1}(R) \cdot \ldots \cdot \lambda_{1,n}^{-k_n}(R) = \lambda_1^{-K}(R), \quad K = (k_1, \dots, k_n),$$

where $\lambda_1(R) := (\lambda_{1,1}(R), \dots, \lambda_{1,n}(R)) \in \mathbb{R}^n_+$. Hence,

$$S_{p_0}(z_0, R) = \max\left\{\frac{|F^{(K)}(z)|_p}{K!\mathbf{L}^K(z)}\colon \|K\| \le N, z \in \mathbb{D}^n\left[z_0, p_0 R/(q\mathbf{L}(z_0))\right]\right\} =$$

$$= \max\left\{\frac{|F^{(K)}(z)|_p}{K!\mathbf{L}^K(z_0)}\cdot\frac{\mathbf{L}^K(z_0)}{\mathbf{L}^K(z)}\colon \|K\|\le N, z\in\mathbb{D}^n[z_0,p_0R/(q\mathbf{L}(z_0))]\right\}\le$$

$$\le S^*_{p_0}(z_0,R)\max\{\lambda_1(R)^{-K}\colon \|K\|\le N\}\le S^*_{p_0}(z_0,R)\prod_{j=1}^n(\lambda_{1,j}(R))^{-N}. \tag{2}$$

For all $z\in\mathbb{D}^n[z_0,p_0R/(q\mathbf{L}(z_0))]$ by the definition of $\lambda_{2,j}(R)$, for $K=(k_1,\dots,k_n)$ we have

$$\frac{\mathbf{L}^K(z)}{\mathbf{L}^K(z_0)}=\frac{{l_1}^{k_1}(z)}{{l_1}^{k_1}(z_0)}\cdot\ldots\cdot\frac{{l_n}^{k_n}(z)}{{l_n}^{k_n}(z_0)}\le\lambda_{2,1}^{k_1}(R)\cdot\ldots\cdot\lambda_{2,n}^{k_p}(R)=\lambda_2^K(R), \tag{3}$$

where $\lambda_2(R):=(\lambda_{2,1}(R),\dots,\lambda_{2,n}(R))\in\mathbb{R}^n_+$. Hence, one has:

$$S^*_{p_0}(z_0,R)\le\max\left\{\frac{|F^{(K)}(z)|_p}{K!\mathbf{L}^K(z)}\lambda_2(R)^K\colon \|K\|\le N, z\in\mathbb{D}^n[z_0,p_0R/(q\mathbf{L}(z_0))]\right\}\le$$

$$\le S_{p_0}(z_0,R)\prod_{j=1}^n(\lambda_{2,j}(R))^N. \tag{4}$$

Let $K_{p_0}\in\mathbb{Z}^n_+$, $\|K_{p_0}\|\le N$ and $z_*\in\mathbb{D}^n[z_0,p_0R/(q\mathbf{L}(z_0))]$ be such that

$$S^*_{p_0}(z_0,R)=\frac{|F^{(K_{p_0})}(z_*)|_p}{K_{p_0}!\mathbf{L}^{K_{p_0}}(z_0)}\ . \tag{5}$$

Since by the maximum modulus principle $z_*\in\mathbb{T}^n(z_0,p_0R/(q\mathbf{L}(z_0)))$, therefore $z_*\ne z_0$. We choose $\widetilde{z}=z_0+\dfrac{p_0-1}{p_0}(z_*-z_0)$. Then for $\widetilde{z}=(\widetilde{z}^{(1)},\dots,\widetilde{z}^{(n)})$, $z_0=(z_0^{(1)},\dots,z_0^{(n)})$, $z_*=(z_*^{(1)},\dots,z_*^{(n)})$, $1\le j\le n$ sequentially we have

$$|\widetilde{z}^{(j)}-z_0^{(j)}|=\frac{p_0-1}{p_0}|z_*^{(j)}-z_0^{(j)}|=\frac{p_0-1}{p_0}\frac{p_0r_j}{ql_j(z_0)}, \tag{6}$$

$$|\widetilde{z}^{(j)}-z_*^{(j)}|=|z_0^{(j)}+\frac{p_0-1}{p_0}(z_*^{(j)}-z_0^{(j)})-z_*^{(j)}|=\frac{1}{p_0}|z_0^{(j)}-z_*^{(j)}|=\frac{r_j}{ql_j(z_0)}. \tag{7}$$

We obtain $\widetilde{z}\in\mathbb{D}^n[z_0,(p_0-1)R/(q(R)\mathbf{L}(z_0))]$ and thus $S^*_{p_0-1}(z_0,R)\ge\dfrac{|F^{(K_{p_0})}(\widetilde{z})|_p}{K_{p_0}!\mathbf{L}^{K_{p_0}}(z_0)}$.

Remark that

$$\frac{d}{dt}\|F^{(K_{p_0})}(\widetilde{z}+t(z_*-\widetilde{z}))\|\le\sum_{j=1}^n\Big(|z_*^{(j)}-\widetilde{z}^{(j)}|\cdot\|F^{(K_{p_0}+\mathbf{e}_j)}(\widetilde{z}+t(z_*-\widetilde{z}))\|\Big)$$

Then, from (5) by the mean value theorem we have

$$0\le S^*_{p_0}(z_0,R)-S^*_{p_0-1}(z_0,R)\le$$

$$\le\frac{|F^{(K_p)}(z_*)|_p-|F^{(K_{p_0})}(\widetilde{z})|_p}{K!\mathbf{L}^{K_{p_0}}(z_0)}=\frac{1}{K_{p_0}!\mathbf{L}^{K_{p_0}}(z_0)}\int_0^1\frac{d}{dt}|F^{(K_{p_0})}(\widetilde{z}+t(z_*-\widetilde{z}))|_p dt\le$$

$$\le\frac{1}{K_{p_0}!\mathbf{L}^{K_{p_0}}(z_0)}\int_0^1\sum_{j=1}^n\Big(|z_*^{(j)}-\widetilde{z}^{(j)}|\cdot|F^{(K_{p_0}+e_j)}(\widetilde{z}+t(z_*-\widetilde{z}))|_p\Big)dt=$$

$$= \frac{1}{K_{p_0}!\mathbf{L}^{K_{p_0}}(z_0)} \sum_{j=1}^{n} \Big(|z_*^{(j)} - \widetilde{z}^{(j)}| \cdot |F^{(K_{p_0}+e_j)}(\widetilde{z} + t^*(z_* - \widetilde{z}))|_p\Big), \tag{8}$$

where $0 \le t^* \le 1$, and $(\widetilde{z} + t^*(z_* - \widetilde{z})) \in \mathbb{D}^n[z_0, p_0R/(q\mathbf{L}(z_0))]$.
For $z \in \mathbb{D}^n[z_0, p_0R/(q\mathbf{L}(z_0))]$ and $J = (j_1, \ldots, j_n) \in \mathbb{Z}_+^n \colon \|J\| \le N+1$, by the definition of the numbers $N = N(F, \mathbf{L})$ and $\lambda_{2,j}\big(p_0R/q\big)$, we have

$$\frac{|F^{(J)}(z)|_p}{J!\mathbf{L}^J(z_0)} = \frac{|F^{(J)}(z)|_p}{J!\mathbf{L}^J(z_0)} \cdot \frac{\mathbf{L}^J(z)}{\mathbf{L}^J(z)} \le \frac{|F^{(J)}(z)|_p}{J!\mathbf{L}^J(z)} \max\left\{\frac{\mathbf{L}^J(z)}{\mathbf{L}^J(z_0)} \colon \|J\| \le N+1\right\} \le$$
$$\le \max\left\{\frac{|F^{(K)}(z)|_p}{K!\mathbf{L}^K(z)} \colon \|K\| \le N\right\} \cdot \prod_{j=1}^{n} \big(\lambda_{2,j}\big(p_0R/q\big)\big)^{N+1} \le$$
$$\le \prod_{j=1}^{n} \big(\lambda_{2,j}(R)\big)^{N+1} \cdot \max\left\{\frac{|F^{(K)}(z)|_p}{K!\mathbf{L}^K(z)} \colon \|K\| \le N\right\} =$$
$$= \prod_{j=1}^{n} \big(\lambda_{2,j}(R)\big)^{N+1} \cdot S_{p_0}(z_0, R) \le \prod_{j=1}^{n} \big(\lambda_{2,j}(R)\big)^{N+1} \cdot S^*_{p_0}(z_0, R) \cdot \prod_{j=1}^{n} \big(\lambda_{1,j}(R)\big)^{-N}.$$

Hence, and from (8), (6) we obtain

$$0 \le S^*_{p_0}(z_0, R) - S^*_{p_0-1}(z_0, R) \le$$
$$\le \sum_{j=1}^{n} \Big(|z_*^{(j)} - \widetilde{z}^{(j)}| \cdot \frac{(K_{p_0}+e_j)!\mathbf{L}^{K_{p_0}+e_j}(z_0)}{K_{p_0}!\mathbf{L}^{K_{p_0}}(z_0)} \cdot \frac{\|F^{(K_{p_0}+e_j)}(\widetilde{z} + t^*(z_* - \widetilde{z}))\|}{(K_{p_0}+e_j)!\mathbf{L}^{K_{p_0}+e_j}(z_0)}\Big) \le$$
$$\le \prod_{j=1}^{n} \Big(\big(\lambda_{2,j}(R)\big)^{N+1}\big(\lambda_{1,j}(R)\big)^{-N}\Big) S^*_{p_0}(z_0, R) \times \sum_{j=1}^{n} |z_*^{(j)} - \widetilde{z}^{(j)}| \langle e_j, K_{p_0} + e_j\rangle \mathbf{L}^{e_j}(z_0).$$

From (7) we have $\sum_{j=1}^{n} |z_*^{(j)} - \widetilde{z}^{(j)}| \langle e_j, K_{p_0} + e_j\rangle \mathbf{L}^{e_j}(z_0) = \frac{1}{q(R)} \sum_{j=1}^{n} \langle e_j, K_{p_0} + e_j\rangle R^{e_j}$, but $\sum_{j=1}^{n} \langle e_j, K_{p_0} + e_j\rangle R^{e_j} \le (N+1)\sum_{j=1}^{n} R^{e_j} = (N+1)\|R\|$. Therefore, using the of choice of $q(R)$ we get

$$S^*_{p_0}(z_0, R) - S^*_{p_0-1}(z_0, R) \le \prod_{j=1}^{n} \Big(\big(\lambda_{2,j}(R)\big)^{N+1}\big(\lambda_{1,j}(R)\big)^{-N}\Big) \frac{S^*_{p_0}(z_0, R)}{q(R)} (N+1)\|R\| \le \frac{S^*_{p_0}(z_0, R)}{2}.$$

It follows that $S^*_{p_0}(z_0, R) \le 2S^*_{p_0-1}(z_0, R)$ and in view of (2) and (4) one has

$$S_{p_0}(z_0, R) \le 2\prod_{j=1}^{n} (\lambda_{1,j}(R))^{-N} S^*_{p_0-1}(z_0, R) \le 2\prod_{j=1}^{n} \Big((\lambda_{1,j}(R))^{-N}(\lambda_{2,j}(R))^{N}\Big) S_{p_0-1}(z_0, R).$$

Then we consequently obtain

$$S_q(z_0, R) \le 2^q \prod_{j=1}^{n} \Big((\lambda_{1,j}(R))^{-N}(\lambda_{2,j}(R))^{N}\Big)^q S_0(z_0, R),$$
$$\max\left\{\frac{|F^{(K)}(z)|_p}{K!\mathbf{L}^K(z)} \colon \|K\| \le N, z \in \mathbb{D}^p\left[z_0, R/\mathbf{L}(z_0)\right]\right\} =$$

$$= \max\left\{\frac{|F^{(K)}(z)|_p}{K!\mathbf{L}^K(z)} \colon \|K\| \le N, z \in \mathbb{D}^p\left[z_0, qR/(q\mathbf{L}(z_0))\right]\right\} = S_q(z_0, R) \le$$

$$= 2^q \prod_{j=1}^{n} \Big((\lambda_{1,j}(R))^{-N}(\lambda_{2,j}(R))^N\Big)^q \max\left\{\frac{|F^{(K)}(z_0)|_p}{K!\mathbf{L}^K(z_0)} \colon \|K\| \le N\right\}. \tag{9}$$

This inequality implies (1) with $p_0 = 2^q \prod_{j=1}^n \Big((\lambda_{1,j}(R))^{-N}(\lambda_{2,j}(R))^N\Big)^q$ and some K_0 such that $\|K_0\| \le N$. The necessity of condition (1) is proved.

Sufficiency. Assume that for every $R \in \mathbb{R}^n_+$ there exist $n_0 \in \mathbb{Z}_+$, $p_0 > 1$, such that for every $z_0 \in \mathbb{C}^n_+$ and for some $K_0 \in \mathbb{Z}^n_+$, $\|K_0\| \le n_0$), inequality (1) holds. By Cauchy's integral formula we have $(\forall z_0 \in \mathbb{C}^n)$, $(\forall K \in \mathbb{Z}^n_+)$, $(\forall S \in \mathbb{Z}^n_+)$:

$$\frac{F^{(K+S)}(z_0)}{S!} = \frac{1}{(2\pi i)^p} \int_{\mathbb{T}^p(z_0, R/\mathbf{L}(z_0))} \frac{F^{(K)}(z)}{(z-z_0)^{S+\mathbf{1}}} dz.$$

Therefore,

$$\frac{|F^{(K+S)}(z_0)|_p}{S!} \le \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n(z_0, R/\mathbf{L}(z_0))} \frac{|F^{(K)}(z)|_p}{|(z-z_0)^{S+\mathbf{1}}|} |dz| \le \int_{\mathbb{T}^n(z_0, R/\mathbf{L}(z_0))} |F^{(K)}(z)|_p \frac{\mathbf{L}^{S+\mathbf{1}}(z_0)}{(2\pi)^n R^{S+\mathbf{1}}} |dz|,$$

where $|dz| = |dz_1| \cdot \ldots \cdot |dz_n|$. Hence, in view of (1), we obtain that

$$\frac{|F^{(K+S)}(z_0)|_p}{S!} \le \frac{p_0 K!}{K_0!(2\pi)^n} |F^{(K_0)}(z_0)|_p \frac{\mathbf{L}^{S+\mathbf{1}}(z_0)}{\mathbf{L}^{K_0}(z_0) R^{S+\mathbf{1}}} \int_{\mathbb{T}^n(z_0, R/\mathbf{L}(z_0))} \mathbf{L}^K(z) |dz|.$$

But, for all $z \in \mathbb{D}^n\left[z_0, R/\mathbf{L}(z_0)\right]$ by the definition of $\lambda_{2,j}(R)$ we have

$$\mathbf{L}^K(z) = \mathbf{L}^K(z_0) \cdot \frac{\mathbf{L}^K(z)}{\mathbf{L}^K(z_0)} = \mathbf{L}^K(z_0) \cdot \frac{{l_1}^{k_1}(z)}{{l_1}^{k_1}(z_0)} \cdot \ldots \cdot \frac{{l_n}^{k_n}(z)}{{l_n}^{k_n}(z_0)} \le$$

$$\le \mathbf{L}^K(z_0) \lambda_{2,1}^{k_1}(R) \cdot \ldots \cdot \lambda_{2,n}^{k_n}(R) = \mathbf{L}^K(z_0) \lambda_2^K(R), \quad K = (k_1, \ldots, k_p),$$

Hence,

$$\frac{|F^{(K+S)}(z_0)|_p}{(K+S)!\mathbf{L}^{K+S}(z_0)} \le \frac{|F^{(K_0)}(z_0)|_p}{K_0!\mathbf{L}^{K_0}(z_0)} \frac{p_0 K! S!}{(K+S)!} \frac{\lambda_2^K(R)}{R^S}. \tag{10}$$

We note that $\frac{K!S!}{(K+S)!} \le 1$ $(\forall K, S \in \mathbb{Z}^n_+)$, and $R^S \to +\infty$ as $\|S\| \to +\infty$ for every $R \in (1, +\infty)^n$. Therefore, for each fixed $R \in (1, +\infty)^n$ and every $K \in \mathbb{Z}^n_+$, $\|K\| \le n_0$, there exists $s_0 \in \mathbb{N}$ such that for every $S \in \mathbb{Z}^n_+$, $\|S\| \ge s_0$, the inequality

$$\frac{p_0 K! S!}{(K+S)!} \frac{\lambda_2^K(R)}{R^S} \le 1$$

holds. Then, in view of (10), one has

$$\frac{|F^{(K+S)}(z_0)|_p}{(K+S)!\mathbf{L}^{K+S}(z_0)} \le \frac{|F^{(K_0)}(z_0)|_p}{K_0!\mathbf{L}^{K_0}(z_0)}$$

for all K, S such that $\|K_0\| \le n_0, \|S\| \ge s_0$. It implies that $\forall z \in \mathbb{C}^n$ $\forall J \in \mathbb{Z}^n_+$:

$$\frac{|F^J(z)|_p}{J!\mathbf{L}^J(z)} \le \max\left\{\frac{|F^K(z)|_p}{K!\mathbf{L}^K(z)} \colon K \in \mathbb{Z}^p_+, \|K\| \le s_0 + n_0\right\}.$$

where s_0 and n_0 do not depend on z_0. Then the entire vector-valued function F has bounded $\mathbf{L}$-index in joint variables $N(F, \mathbf{L}) \le s_0 + n_0$. The proof of theorem is complete. □

Theorem 1 implies the following corollary.

Corollary 1. *Let* $\mathbf{L} \in \mathbb{Q}^p$ *and* $\|\cdot\|_0$ *be some norm in* $\mathbb{C}^p$. *An entire vector-function* $F\colon \mathbb{C}^p \to \mathbb{C}^p$ *has bounded* $\mathbf{L}$*-index in joint variables in* sup*-norm if and only if it has bounded* $\mathbf{L}$*-index in joint variables in the norm* $\|\cdot\|_0$.

Proof. Recall that ([12]) if $\|\cdot\|_1$ and $\|\cdot\|_2$ are two norms in $\mathbb{C}^p$, then there exist constants $C_1, C_2 \in (0, +\infty)$ such that $C_1\|w\|_1 \le \|w\|_2 \le C_2\|w\|_1$ for every $w \in \mathbb{C}^p$. Thus, for all $K \in \mathbb{Z}_+^p$ and for all $z \in \mathbb{C}^p$ we obtain

$$C_1\|F^{(K)}(z)\| \le \|F^{(K)}(z)\|_0 \le C_2\|F^{(K)}(z)\|,$$

where $\|\cdot\|$ is the sup-norm. Using the given inequalities and repeating arguments from Theorem 1 for the case of the Euclidean norm we can verify the equivalence of these norms for vector-functions having bounded $\mathbf{L}$-index in joint variables. □

From Corollary 1, in particular, it follows that instead of the sup-norm $\|A\| = \max\limits_{1\le j\le p} |a_j|$ one can consider in Theorem 1 the Euclidean norm $\|A\|_E = \sqrt{|a_1|^2 + \ldots + |a_p|^2}$, where $A = (a_1, \ldots, a_p) \in \mathbb{C}^p$.

REFERENCES

1. V.P. Baksa, *Analytic vector-valued functions in the unit ball having bounded L-index in joint variables*, Carpathian Math. Publ., **11** (2019), №2, 213–227. doi:10.15330/cmp.11.2.213-227
2. V.P. Baksa, A.I. Bandura, O.B. Skaskiv, *Analogs of Fricke's theorems for analytic vector-valued functions in the unit ball having bounded* $\mathbf{L}$*-index in joint variables*, Proceedings of IAMM of NASU, **33** (2019), 16–26. doi: 10.37069/1683-4720-2019-33-1
3. V. Baksa, A. Bandura, O. Skaskiv, *Growth estimates for analytic vector-valued functions in the unit ball having bounded L-index in joint variables*, Constructive Math. Analysis, **3** (2020), №1, 9–19. doi: 10.33205/cma.650977
4. Bandura A., Skaskiv O. *Boundedness of the L-index in a direction of entire solutions of second order partial differential equation*, Acta Comment. Univ. Tartu. Math., 2018, **22**, №2: 223–234. doi: 10.12697/ACUTM.2018.22.18
5. A. Bandura, O. Skaskiv, L. Smolovyk, *Slice holomorphic solutions of some directional differential equations with bounded L-index in the same direction*, Demonstr. Math., **52** (2019), №1, 482–489. doi: 10.1515/dema-2019-0043.
6. A. Bandura, O. Skaskiv, *Sufficient conditions of boundedness of* $\mathbf{L}$*-index and analog of Hayman's Theorem for analytic functions in a ball*, Stud. Univ. Babeş-Bolyai Math., **63** (2018), №4, 483–501. doi:10.24193/subbmath.2018.4.06
7. A. Bandura, O. Skaskiv, Entire functions of several variables of bounded index, Lviv: Publisher I.E.Chyzhykov, 2016, 128 p.
8. A. Bandura, O. Skaskiv, *Asymptotic estimates of entire functions of bounded* $\mathbf{L}$*-index in joint variables*, Novi Sad J. Math., **48** (2018) №1, 103–116. doi: 10.30755/NSJOM.06997
9. A. Bandura, N. Petrechko, O. Skaskiv, *Maximum modulus in a bidisc of analytic functions of bounded* $\mathbf{L}$*-index and an analogue of Hayman's theorem*, Mat. Bohemica, **143** (2018), №4, 339–354. doi: 10.21136/MB.2017.0110-16
10. A.I. Bandura, O.B. Skaskiv, V.L. Tsvigun, *Some characteristic properties of analytic functions in* $\mathbb{D}\times\mathbb{C}$ *of bounded* $\mathbf{L}$*-index in joint variables*, Bukovyn. Mat. Zh., **6** (2018), №1-2, 21–31.
11. A. Bandura, O. Skaskiv, *Analog of Hayman's Theorem and its application to some system of linear partial differential equations*, J. Math. Phys., Anal., Geom., **15** (2019), №2, 170–191. doi: 10.15407/mag15.02.170

12. Lelong P., Gruman L. Entire functions of several complex variables, Springer Verlag, Berlin-Heidelberg, New York-Tokyo, 1986.
13. W.K. Hayman, *Differential inequalities and local valency*, Pacific J. Math., **44** (1973), №1, 117–137.
14. B. Lepson, *Differential equations of infinite order, hyperdirichlet series and entire functions of bounded index,* Proc. Sympos. Pure Math., (1968) **2**, 298–307.
15. R.F. Patterson, F. Nuray, *A characterization of holomorphic bivariate functions of bounded index*, Math. Slov., **67** (2017), №3, 731–736. doi: 10.1515/ms-2017-0005
16. F. Nuray, R.F. Patterson, *Vector-valued bivariate entire functions of bounded index satisfying a system of differential equations*, Mat. Stud., **49** (2018), №1, 67–74. doi: 10.15330/ms.49.1.67-74
17. M. Sheremeta, Analytic functions of bounded index, Lviv: VNTL Publishers, 1999, 141 p.
18. M. Sheremeta, Geometric properties of analytic solutions of differential equations, Lviv: Publisher I. E. Chyzhykov, 2019, 164 p.

Ivan Franko National University of Lviv
Lviv, Ukraine
vitalinabaksa@gmail.com

Ivano-Frankivsk National Technical University of Oil and Gas
Ivano-Frankivsk, Ukraine
andriykopanytsia@gmail.com

Received 12.03.2020
Revised 20.08.2020