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GEODESIC MAPPINGS OF QUASI-EINSTEIN SPACES WITH A CONSTANT SCALAR CURVATURE

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In this paper we study a special type of pseudo-Riemannian spaces – quasi-Einstein spaces of constant scalar curvature. These spaces are generalizations of known Einstein spaces. We obtained a linear form of the basic equations of the theory of geodetic mappings for these spaces. The studies are conducted locally in tensor form, without restrictions on the sign and signature of the metric tensor.

1. Introduction. E. Beltrami was the first to consider the question of geodesic mapping of a surface V_2 into a surface $\bar{E}_2$ as early as 1865 ([1]). He sought a solution for classical problems of cartography known since Lagrange ([15]). In 1869 U. Dini ([2]) posed a general problem of a possibility of geodesic mapping for a given surface V_2 into $\bar{V}_2$. Actually he solved this problem for Riemannian spaces, however he did it in such a complex way, that the solution was improved since then on many occasions. In 1896 T. Levi-Civita ([16]) proposed a particular formulation of the problem (implied by dynamics equations) and obtained main equations in tensor form ([5]).

Thereafter tensor methods took the leading role in differential geometry. H. Weyl, L. P. Eisenhart, V. F. Kagan, G. I. Kruchkovich, A. S. Solodovnikov and others developed a coherent theory of geodesic mappings of pseudo-Riemannian spaces that was invariant in relation to the choice of coordinate system.

M. S. Syniukov pushed the research further by reduction of the problem to a study of linear system of differential equations ([18]).

The linear form of basic equations of theory of geodesic mappings was simplified and there was a solution found for the problem of cardinalities distribution for a geodesic class of a given space ([12]).

Significant progress has been achieved in the study of special pseudo-Riemannian spaces, Einstein spaces in particular ([11, 17]).

It appeared that four-dimensional Einstein spaces that differs from spaces of a constant curvature, do not permit non-trivial geodesic mappings. This fact underlined the necessity of a research on more general classes of spaces. The latter were built by adding to the internal objects (Ricci tensor, Einstein tensor) both constructions made of internal objects ([13, 14]), and some special vector fields ([7, 8]).

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In this paper, following [6], we study spaces in which the Einstein tensor deviates from zero by some bivector.

2. Basic equations of the theory of geodesic mappings. The one-to-one correspondence between the points of pseudo-Riemannian spaces V_n with the metric tensor g_{ij} and $\overline{V}_n$ with a metric tensor $\bar{g}_{ij}$ is called a geodesic mapping if any geodesic line in V_n is mapped into a geodesic line in $\overline{V}_n$.

If pseudo-Riemannian spaces V_n and $\overline{V}_n$ allow bijective geodesic mapping, we call them spaces that are in geodesic correspondence, or spaces that belong to the same geodesic class.

A necessary and sufficient condition [16] for the pseudo-Riemannian spaces V_n and $\overline{V}_n$ to allow geodetic mapping on each other is

$$\bar{\Gamma}^h_{ij} = \Gamma^h_{ij} + \varphi_i \delta^h_j + \varphi_j \delta^h_i, \tag{1}$$

or, considering the covariant constancy of the metric tensor

$$\bar{g}_{ij},_k = 2\varphi_k \bar{g}_{ij} + \varphi_i \bar{g}_{jk} + \varphi_j \bar{g}_{ik}, \tag{2}$$

where φ_i is some necessary gradient vector, $\Gamma^h_{ij}, \bar{\Gamma}^h_{ij}$ are Christoffel symbols V_n and $\bar{V}_n$ respectively; δ^h_i are Kronecker symbols; comma ",“ is the sign of the covariant derivative in respect to connectivity of V_n.

Equations (1) and (2) are equivalent, necessary, and sufficient conditions for pseudo-Riemannian spaces V_n and $\overline{V}_n$ to be in geodesic correspondence.

A necessary condition for geodesic mapping is given by the equations:

$$\bar{R}^h_{ijk} = R^h_{ijk} + \varphi_{ij}\delta^h_k - \varphi_{ik}\delta^h_j, \quad \bar{R}_{ij} = R_{ij} + (n-1)\varphi_{ij}, \tag{3}$$

where $\varphi_{ij} = \varphi_{i,j} - \varphi_i \varphi_j$, R^h_{ijk}, R_{ijk} are Riemann and Ricci tensors.

A geodesic mapping that differs from homothetic is called non-trivial.

The given pseudo-Riemannian space V_n permits a non-trivial geodesic mapping only in the case when the system of differential equations has a solution in respect to the tensor $a_{ij} = a_{ji} \neq c g_{ij}$ and the vector $\lambda_i = \lambda_{,i} \neq 0$. It is a necessary and sufficient condition.

The linear form of the basic equations of the theory of geodesic mappings can be written down as follows ([18, p.121])

$$a_{ij,k} = \lambda_i g_{jk} + \lambda_j g_{ik}. \tag{4}$$

$$n\lambda_{i,j} = \mu g_{ij} + a_{\alpha i} R^\alpha_j - a_{\alpha\beta} R^{\alpha}_{\cdot\, ij\, \cdot}{}^{\beta}, \tag{5}$$

here $\mu = \lambda_{\alpha,\beta} g^{\alpha\beta}$; $R^i_j = R_{\alpha j} g^{\alpha i}$; $R^{h}{}_{ij}{}^{k} = R^h_{ij\alpha} g^{\alpha}{}_{\cdot}{}^{k}$.

From the latter we will have ([18, p.123]):

$$(n-1)\mu_{,i} = 2(n+1)\lambda_\alpha R^\alpha_i + a_{\alpha\beta}(2R^{\alpha}_{\cdot\, i,\, \cdot}{}^{\beta} - R^{\alpha\beta}{}_{,i}). \tag{6}$$

Solutions (2) and (4) are connected by relations

$$a_{ij} = e^{2\varphi} \bar{g}^{\alpha\beta} g_{\alpha i} g_{\beta j}; \quad \lambda_i = -e^{2\varphi} \bar{g}^{\alpha\beta} g_{\alpha i} \varphi_\beta.$$

The system of equations (4), (5) and (6) gives a fundamental possibility to answer the question: does a given pseudo-Riemannian space V_n allow geodesic mapping to pseudo-Riemannian space $\bar{V}_n$. The question is reduced to a study of integrability conditions of these differential equations and their differential extensions ([12]).

The purpose of our work is to obtain the form of basic equations of the theory of geodesic mappings for quasi-Einstein spaces.

3. Basic equations of the theory of geodesic mappings of quasi-Einstein spaces. Let us consider a geodesic mapping of quasi-Einstein spaces, namely pseudo-Riemannian spaces $V_n (n > 2)$ which satisfy the following condition

$$R_{ij} = \frac{R}{n} g_{ij} + U_i U_j, \tag{7}$$

where U_i is a gradient vector by definition, i.e. $U_i = U_{,i} = \partial_i U$. It follows from the definition that the vector U_i is, by necessity, an isotropic vector. Given (7), equation ([18, p.138])

$$a_{\alpha l} R^{\alpha}_{h} - a_{\alpha k} R^{\alpha}_{l} = 0,$$

will take the form $U_l U^{\alpha} a_{\alpha i} = U_i a_{\alpha l} U^{\alpha}$.

From the last equality we have

$$U^{\alpha} a_{\alpha i} = \rho U_i, \tag{8}$$

where $\rho \overset{def}{=} a_{\alpha\beta} U^{\alpha} \xi^{\beta}$, ξ^i is some vector such that $U_{\alpha} \xi^{\alpha} = 1$.

Thus, we are proved

Theorem 1. *If quasi-Einstein space V_n permits non-trivial geodesic mapping, then the vector U_i is the eigenvector of the tensor matrix a_{ij}.*

Let us prove the following theorem.

Theorem 2. *If quasi-Einstein space V_n permits non-trivial geodesic mapping, then the vectors U_i and λ_i are mutually orthogonal, that is $U^{\alpha} \lambda_{\alpha} = 0$.*

Proof. Differentiating (8) with respect to (4) we obtain

$$U^{\alpha}{}_{,j} a_{\alpha i} + U^{\alpha} \lambda_{\alpha} g_{ij} + \lambda_i U_j = \rho_{,j} U_i + \rho U_{i,j}. \tag{9}$$

Because of the isotropy of the vector U_i, by multiplying (9) on it and contracting it, we have $2U^{\alpha} \lambda_{\alpha} U_i = 0$, since U_i is not a zero vector, then the theorem is proved.

Let us now consider the question about non-trivial geodesic mapping of quasi-Einstein spaces of constant scalar curvature. □

Let us prove the following theorem.

Theorem 3. *If the quasi-Einstein space of constant scalar curvature allows non-trivial geodesic mapping, the vector λ_i satisfies the conditions*

$$\lambda_{\alpha j,}{}^{\alpha} = \tau \lambda_j, \tag{10}$$

here $\lambda_{i\alpha,}{}^{\alpha} = \lambda_{i,\alpha}{}^{\alpha} = \lambda_{i,\alpha\beta} g^{\alpha\beta}$, and τ is some invariant.

Proof. Differentiating

$$a_{\alpha i} R^{\alpha}_{jkl} + a_{\alpha j} R^{\alpha}_{ikl} = \lambda_{li} g_{jk} + \lambda_{lj} g_{ik} - \lambda_{kj} g_{il} - \lambda_{ki} g_{jl}, \tag{11}$$

where $\lambda_{ij} = \lambda_{i,j}$, according to (4), we obtain

$$\lambda_\alpha R^\alpha_{jkl} g_{im} + \lambda_i R_{mjkl} + \lambda_\alpha R^\alpha_{ikl} g_{jm} + \lambda_j R_{mikl} + a_{\alpha i} R^\alpha_{jkl,m} + a_{\alpha j} R^\alpha_{ikl,m} = \\ = \lambda_{li,m} g_{jk} + \lambda_{lj,m} g_{ik} - \lambda_{ki,m} g_{jl} - \lambda_{kj,m} g_{il}.$$

Contracting the latter by l and m, we have

$$\lambda_\alpha R^\alpha_{jki} + \lambda_\alpha R^\alpha_{ikj} + \lambda_i R_{jk} + \lambda_j R_{ik} + a^\alpha_i R^\beta_{kj\alpha,\beta} + a^\alpha_j R^\beta_{ki\alpha,\beta} = \\ = \lambda_{\alpha i,}{}^\alpha g_{jk} + \lambda_{\alpha j,}{}^\alpha g_{ik} - \lambda_{ki,j} - \lambda_{kj,i}.$$

Given that $R^\alpha_{ijk,\alpha} = R_{ij,k} - R_{ik,j}$ and (7), we obtain

$$\lambda_\alpha R^\alpha_{jki} + \lambda_\alpha R^\alpha_{ikj} + \lambda_i R_{jk} + \lambda_j R_{ik} + U_j(\rho_k U_i + \rho U_{i,k} - \lambda_i U_k) - \rho U_i U_{k,j} + \\ + U_i(\rho_k U_j + \rho U_{j,k} - \lambda_j U_k) - \rho U_j U_{k,i} = \lambda_{\alpha i,}{}^\alpha g_{jk} + \lambda_{\alpha j,}{}^\alpha g_{ik} - \lambda_{ki,j} - \lambda_{kj,i}.$$

Or, just like that,

$$\lambda_\alpha R^\alpha_{jki} + \lambda_\alpha R^\alpha_{ikj} + \lambda_i R_{jk} + \lambda_j R_{ik} + U_j(\rho_k U_i - \lambda_i U_k) + U_i(\rho_k U_j - \lambda_j U_k) = \\ = \lambda_{\alpha i,}{}^\alpha g_{jk} + \lambda_{\alpha j,}{}^\alpha g_{ik} - \lambda_{ki,j} - \lambda_{kj,i}.$$

Alternating the last equality by j, k, we obtain

$$4\lambda_\alpha R^\alpha_{ikj} + 2U_j U_i \rho_k - 2U_i U_k \rho_j + \frac{R}{n}(\lambda_j g_{ik} - \lambda_k g_{ji}) = \lambda_{\alpha j,}{}^\alpha g_{ik} - \lambda_{\alpha k,}{}^\alpha g_{ij}. \qquad (12)$$

Multiplying (12) by λ^i and contracting by i, we get

$$\lambda_{\alpha j,}{}^\alpha \lambda_k - \lambda_{\alpha k,}{}^\alpha \lambda_j = 0.$$

This implies (10), where τ is some invariant such that $\tau = \lambda_{\beta\alpha,}{}^\alpha \eta^\beta$; and η^i is a vector, which satisfies the condition $\lambda_\alpha \eta^\alpha = 1$. □

Given (10), equation (12) takes the form

$$4\lambda_\alpha R^\alpha_{ikj} + 2U_j U_i \rho_k - 2U_i U_k \rho_j + \left(\frac{R}{n} - \tau\right)(\lambda_j g_{ik} - \lambda_k g_{ij}) = 0. \qquad (13)$$

Multiplying (11) by λ^l, and contracting by l with respect to (13), we obtain

$$2a^\alpha_i \rho_\alpha U_k U_j - 2\rho\rho_j U_k U_i + \left(\frac{R}{n} - \tau\right)(\lambda_j a_{ik} - \lambda_\alpha a^\alpha_i g_{jk}) + 2a^\alpha_j \rho_\alpha U_k U_i - 2\rho\rho_i U_k U_j + \\ + \left(\frac{R}{n} - \tau\right)(\lambda_i a_{jk} - \lambda_\alpha a^\alpha_j g_{ik}) = 4\lambda^\alpha \lambda_{\alpha i} g_{jk} + 4\lambda^\alpha \lambda_{\alpha j} g_{ik} - 4\lambda_{ki}\lambda_j - 4\lambda_{kj}\lambda_i. \qquad (14)$$

Let us alternate the last equality by j and k. Then we replace the indices $i \longleftrightarrow k$ in the resulting expression and summarize the result with (14). We have

$$2(a^\alpha_i \rho_\alpha - \rho\rho_i) U_k U_j + \lambda_i \left(\left(\frac{R}{n} - \tau\right) a_{jk} + 4\lambda_{kj}\right) = (4\lambda^\alpha \lambda_{\alpha i} + \left(\frac{R}{n} - \tau\right) \lambda_\alpha a^\alpha_i) g_{jk}. \qquad (15)$$

We contract with g^{jk}, then

$$4\lambda^{\alpha}\lambda_{\alpha i} + \left(\frac{R}{n} - \tau\right)\lambda_{\alpha}a_i^{\alpha} = 4\mu\lambda_i,$$

where

$$4\mu = \frac{1}{n}\left(\left(\frac{R}{n} - \tau\right)a_{\alpha\beta} + 4\lambda_{\alpha\beta}\right)g^{\alpha\beta}.$$

Given this, we write (15) in the form

$$2(a_i^{\alpha}\rho_{\alpha} - \rho\rho_i)U_kU_j + \lambda_i\left(\left(\frac{R}{n} - \tau\right)a_{jk} + 4\lambda_{kj} - 4\mu\, g_{kj}\right) = 0.$$

Contracting the latter equality with η^i, we obtain

$$\left(\frac{R}{n} - \tau\right)a_{jk} + 4\lambda_{kj} - 4\mu\, g_{kj} - 4\overset{1}{c}\,U_kU_j = 0. \tag{16}$$

Here

$$2(a^{\alpha}{}_{\beta}\rho_{\alpha} - \rho\rho_{\beta})\eta^{\beta} \overset{def}{=} -4\overset{1}{c}.$$

It is easy to see that

$$\tau = \frac{R(n+3)}{n(n-1)}.$$

And then (16) will take the final form

$$\lambda_{kj} = \mu g_{kj} + \frac{R}{n(n-1)}a_{kj} + \overset{1}{c}\,U_kU_j. \tag{17}$$

Differentiating (17), we have

$$\lambda_{i,jk} = \mu_{,k}g_{ij} + \frac{R}{n(n-1)}(\lambda_i g_{jk} + \lambda_j g_{ik}) + \overset{1}{c}_{,k}\,U_iU_j + \overset{1}{c}\,U_{i,k}U_j + \overset{1}{c}\,U_iU_{j,k}.$$

Contracting by i, j, we obtain

$$g^{\alpha\beta}\lambda_{\alpha,\beta k} = n\mu_{,k} + \frac{2R}{n(n-1)}\lambda_k.$$

After using the Ricci identity for quasi-Einstein spaces

$$g^{\alpha\beta}(\lambda_{\alpha,\beta k} - \lambda_{\alpha,k\beta}) = \frac{R}{n}\lambda_k,$$

we get

$$\mu_{,i} = \frac{2R}{n(n-1)}\lambda_i. \tag{18}$$

Thus, the theorem is true

Theorem 4. *If quasi-Einstein space V_n of constant scalar curvature permits non-trivial geodesic mappings, then conditions* (17), (18) *are satisfied.*

Conclusions. We defined a form of a system of basic equations for geodesic mappings of quasi-Einstein spaces.

The developed methods of research can be applied in the theory of conformal mappings ([3]) and in the theory of holomorphically projective mappings of Kählerian spaces ([4, 9]).

A further research is needed in order to shed new light on the pseudo-Riemannian spaces that result from geodesic mapping of a quasi-Einstein space.

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