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FINITARY APPROXIMATIONS OF COARSE STRUCTURES

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A coarse structure $\mathcal{E}$ on a set X is called finitary if, for each entourage $E \in \mathcal{E}$, there exists a natural number n such that $E[x] < n$ for each $x \in X$. By a finitary approximation of a coarse structure $\mathcal{E}'$, we mean any finitary coarse structure $\mathcal{E}$ such that $\mathcal{E} \subseteq \mathcal{E}'$. If $\mathcal{E}'$ has a countable base and $E[x]$ is finite for each $x \in X$ then $\mathcal{E}'$ has a cellular finitary approximation $\mathcal{E}$ such that the relations of linkness on subsets of $(X, \mathcal{E}')$ and $(X, \mathcal{E})$ coincide. This answers Question 6 from [8]: the class of cellular coarse spaces is not stable under linkness. We define and apply the strongest finitary approximation of a coarse structure.

1. Introduction. Given a set X, a family $\mathcal{E}$ of subsets of $X \times X$ is called a *coarse structure* on X if

- each $E \in \mathcal{E}$ contains the diagonal $\triangle_X$, $\triangle_X = \{(x,x) \in X : x \in X\}$;
- if E, $E' \in \mathcal{E}$ then $E \circ E' \in \mathcal{E}$ and $E^{-1} \in \mathcal{E}$, where $E \circ E' = \{(x,y) : \exists z((x,z) \in E,\ \ (z,y) \in E')\}$, $E^{-1} = \{(y,x) : (x,y) \in E\}$;
- if $E \in \mathcal{E}$ and $\triangle_X \subseteq E' \subseteq E$ then $E' \in \mathcal{E}$;
- $\bigcup \mathcal{E} = X \times X$.

A subfamily $\mathcal{E}' \subseteq \mathcal{E}$ is called a *base* for $\mathcal{E}$ if, for every $E \in \mathcal{E}$, there exists $E' \in \mathcal{E}'$ such that $E \subseteq E'$. For $x \in X$, $A \subseteq X$ and $E \in \mathcal{E}$, we denote

$$E[x] = \{y \in X : (x,y) \in E\}, \quad E[A] = \bigcup_{a \in A} E[a], \quad E_A[x] = E[x] \cap A$$

and say that $E[x]$ and $E[A]$ are *balls of radius E around* x and A.

The pair $(X, \mathcal{E})$ is called a *coarse space* [10] or a ballean [7], [9].

For a coarse space $(X, \mathcal{E})$, a subset $B \subseteq X$ is called *bounded* if $B \subseteq E[x]$ for some $E \in \mathcal{E}$ and $x \in X$. The family $\mathcal{B}_{(X,\mathcal{E})}$ of all bounded subsets of $(X, \mathcal{E})$ is called the *bornology* of $(X, \mathcal{E})$. We recall that a family $\mathcal{B}$ of subsets of a set X is a bornology if $\mathcal{B}$ is closed under taking subsets and finite unions, and $\mathcal{B}$ contains all finite subsets of X.

We say that $(X, \mathcal{E})$ is *locally finite,* if each ball $E[x]$ is finite, equivalently, $\mathcal{B}_{(X,\mathcal{E})} = [X]^{<\omega}$.

A coarse space $(X, \mathcal{E})$ is called *finitary,* if for each $E \in \mathcal{E}$ there exists a natural number n such that $|E[x]| < n$ for each $x \in X$.

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Let G be a transitive group of permutations of a set X. We denote by X_G the set X endowed with the coarse structure with the base

$$\{\{(x, gx)\colon g \in F\}\colon F \in [G]^{<\omega},\ id \in F\}.$$

By [4, Theorem 1], for every finitary coarse structure $(X, \mathcal{E})$, there exists a transitive group G of permutations of X such that $(X, \mathcal{E}) = X_G$. For more general results, see [6].

By a *finitary approximation* of a coarse structure $\mathcal{E}'$ on X we mean an arbitrary finitary coarse structure $\mathcal{E}$ on X such that $\mathcal{E} \subseteq \mathcal{E}'$. In particular, the discrete coarse structure $\mathcal{E}_0$ with the base $\{\{(x, y)\colon x, y \in F\} \cup \Delta_X\colon F \in [X]^{<\omega}\}$ is a finitary approximation of any coarse structure on X.

Following [8], we say that two subsets A, B of a coarse space $(X, \mathcal{E})$ are *linked* (write $(A, B) \in \lambda_{(X,\mathcal{E})}$) if either A, B are bounded or there exists $E \in \mathcal{E}$ such that $E[A] \cap B$ is unbounded,

A class $\mathcal{K}$ of coarse space is called λ-*stable* if $(X, \mathcal{E}) \in \mathcal{K}$ and $\lambda_{(X,\mathcal{E})} = \lambda_{(X,\mathcal{E}')}$ imply $(X, \mathcal{E}') \in \mathcal{K}$.

We recall that a coarse space $(X, \mathcal{E})$ is *cellular* if $\mathcal{E}$ has a base consisting of equivalences of X, equivalently, *asdim* $(X, \mathcal{E}) = 0$, see [9, Theorem 3.1.3].

This note is to give the negative ZFC-answer to the following question (see [8, Question 6] and [3, Problem 2.11]): *is the class of cellular coarse spaces λ-stable?* The same answer but under some set-theoretical assumptions involving small cardinals was obtained by Taras Banakh, see [8, Remark 1] and [1, Corollary 5.9].

2. Results.

Theorem 1. *Let $(X, \mathcal{E}')$ be an unbounded locally finite coarse space with a countable base of $\mathcal{E}'$. Then there exists a cellular finitary approximation $\mathcal{E}$ of X such that $\lambda_{(X,\mathcal{E})} = \lambda_{(X,\mathcal{E}')}$.*

Proof. We use the Zorn Lemma to choose a maximal by inclusion cellular finitary approximation $\mathcal{E}$ of $\mathcal{E}'$ such that $\mathcal{E}_0 \subseteq \mathcal{E} \subseteq \mathcal{E}'$, $\mathcal{E}_0$ has the base $\{\{(x, y)\colon x, y \in F\} \bigcup \Delta_X\colon F \in [X]^{<\omega}\}$. Let $\mathcal{P}$ be a base of $\mathcal{E}$ consisting of equivalences, and let $\{E'_n\colon n < \omega\}$ be an increasing symmetric base of $\mathcal{E}'$. Since $(X, \mathcal{E}')$ is locally finite, we have $\lambda_{(X,\mathcal{E})} \subseteq \lambda_{(X,\mathcal{E}')}$.

To prove $\lambda_{(X,\mathcal{E}')} \subseteq \lambda_{(X,\mathcal{E})}$, we take any two unbounded subsets A, B of X such that $(A, B) \in \lambda_{(X,\mathcal{E}')}$, so $E'[A] \cap B$ is unbounded for some $E' \in \mathcal{E}'$, $(E')^{-1} = E'$.

We choose inductively two injective sequences $(a_n)_{n<\omega}$ in A and $(b_n)_{n<\omega}$ in B such that $(a_n, b_n) \in E'$ and

$$E'_{n+1}[\{a_{n+1}, b_{n+1}\}] \cap E'_{n+1}[\{a_0, b_0, \ldots, a_n, b_n\}] = \varnothing. \tag{$*$}$$

We put $M = \{(a_n, b_n), (b_n, a_n)\colon n < \omega\} \cup \Delta_X$ and denote by $\mathcal{M}$ the smallest coarse structure on X such that $M \in \mathcal{M}$ and $\mathcal{E} \subseteq \mathcal{M}$. Since A, B are linked in $\mathcal{M}$, it suffices to show that $\mathcal{E} = \mathcal{M}$. In turn on, by the maximality of $\mathcal{E}$ and $\mathcal{M} \subseteq \mathcal{E}'$, it suffices to verify that $\mathcal{M}$ is cellular.

We take an arbitrary $P \in \mathcal{P}$, a natural number m, put $H = M \cup P$ and show that H^m is contained in some equivalence from $\mathcal{M}$. We choose a natural number n such that $P \subseteq E'_{n+1}$.

If $x \in X$ and $E'_{n+1}[x] \cap E'_{n+1}[\{a_0, b_0, \ldots, a_n, b_n\}] = \varnothing$ then by $(*)$ either $H^m[x] = P[x]$ or $H^m[x]$ is the union of two blocks of P containing some $\{a_k, b_k\}$, $k > n$. We enlarge the obtained partial equivalence of X by the joining the block

$$\bigcup \{P[x] \colon E'_{n+1}[x] \ \cap \ E'_{n+1}[\{a_0, b_0, \dots, a_n, b_n\}] \neq \varnothing\}$$

and get the desired equivalence from $\mathcal{M}$ containing $H^m[x]$. □

Remark 1. For every natural number n, there exists a coarse structure $\mathcal{E}'$ satisfying Theorem 1 such that asdim $(X, \mathcal{E}') = n$. Analogously, there exists $\mathcal{E}'$ such that *asdim* $(X, \mathcal{E}') = \infty$. Theorem 1 shows that the corresponding classes of coarse spaces are not λ-stable.

For a coarse structure $\mathcal{E}$ on X, we denote by $\mathcal{E}_{fin}$ the strongest finitary coarse structure on X such that $\mathcal{E}_{fin} \subseteq \mathcal{E}$. We take the family $\mathcal{F} \in \mathcal{E}$ such that $E = E^{-1}$ and there exists a natural number n such that $|E[x]| < n$ for every $x \in X$. Then $\mathcal{F}$ is a base of $\mathcal{E}_{fin}$.

Proposition 1. *If a coarse structure $\mathcal{E}$ on X is locally finite then $\lambda_{(X,\mathcal{E})} = \lambda_{(X,\mathcal{E}_{fin})}$.*

Proof. Since $\mathcal{E}$ is locally finite, we have $\lambda_{(X,\mathcal{E}_{fin})} \subseteq \lambda_{(X,\mathcal{E})}$. Let A, B be infinite subset of X such that $(A, B) \in \lambda_{(X,E)}$. We take $E \in \mathcal{E}$ such that $E = E^{-1}$ and $E[A] \cap B$ is infinite. We construct inductively a countable subset A' of A and an injective mapping $f \colon A' \longrightarrow B$ such that $(x, f(x)) \in E$ for each $x \in A'$. Then the entourage $\{(x, f(x)) \colon x \in A'\} \cup \Delta_X$ in $\mathcal{E}_{fin}$ shows that A, B are linked in $(X, \mathcal{E}_{fin})$. □

Theorem 2. *Let $(X, \mathcal{E})$ be a locally finite coarse space. Assume that there exists $E \in \mathcal{E}$ such that, for every natural number n, there exists $x \in X$ such that $|E[x]| > n$. Then the coarse space $(X, \mathcal{E}_{fin})$ is not cellular.*

Proof. We choose a sequence $(x_n)_{n<\omega}$ in X such that the subsets $\{E[x_n] \colon n < \omega\}$ are pairwise disjoint and $|E[x_n]| > n$. Let $|E[x_n]| = m_n$. We enumerate $E[x_n] = \{a_{n1}, a_{n2}, \dots, a_{nm_n}\}$ and define a bijection f of X such that, for each n, $f(a_{n1}) = f(a_{n2}), \dots, f(a_{nm_n}) = a_{n1}$, and $f(x) = x$ for each $x \in X \setminus \bigcup_{n<\omega} E[x_n]$. Then the entourage $\{(x, f(x)) \colon x \in X\} \cup \Delta_X$ belongs to $\mathcal{E}_{fin}$ and shows that $\mathcal{E}_{fin}$ is not cellular. Otherwise, there is a equivalence P in $\mathcal{E}_{fin}$ such that each $E[x_n]$ is contained in some block of P contradicting finitarity of $\mathcal{E}_{fin}$. □

Remark 2. If we take cellular $(X, \mathcal{E})$ in Theorem 2 and apply Proposition 1 then also get the negative answer to Question 6 from [7]. In light of Corollary 1 from [2] we may conjecture that asdim $(X, \mathcal{E}_{fin}) = \infty$.

Following [8], we say that two subsets A, B of $(X, \mathcal{E})$ are *close* (write $(A, B) \in \delta_{(X,\mathcal{E})}$) if there exists $E \in \mathcal{E}$ such that $A \subseteq E[B]$, $B \subseteq E[A]$.

Remark 3. We show that Proposition 1 does not hold with $\delta_{(X,\mathcal{E})} = \delta_{(X,\mathcal{E}_fin)}$ in place $\lambda_{(X,\mathcal{E})} = \lambda_{(X,\mathcal{E}_fin)}$. We partition a countable set X into finite subsets $\{X_n \colon n < \omega\}$ such that $|X_n| > n$, denote by E the equivalence defined by this partition and take the smallest coarse structure $\mathcal{E}$ containing E as an entourage. For each $n < \omega$, we pick $x_n \in X_n$, put $A = \{x_n \colon n < \omega\}$, $B = X$. Then $(A, B) \in \delta_{(X,\mathcal{E})}$ but $(A, B) \notin \delta_{(X,\mathcal{E}_fin)}$.

By [1, Theorem 2.8], the strongest finitary coarse structure $\mathcal{F}$ on a set X does not admit a cellular finitary approximation $\mathcal{E}$ such that $\delta_{(X,\mathcal{F})} = \delta_{(X,\mathcal{E})}$.

Question. *Does there exist a non-cellular finitary coarse space $(X, \mathcal{E})$ such that $\mathcal{E}$ has a countable base and $\delta_{(X,\mathcal{E})} = \delta_{(X,\mathcal{E}')}$ for some cellular finitary approximation $\mathcal{E}'$ of $\mathcal{E}$.*

Following [5], we say that a class $\mathcal{K}$ of coarse spaces is a *variety* if $\mathcal{K}$ is closed under operations of taking subspaces $\mathbf{S}$, cartesian products $\mathbf{P}$ and macro-uniform images $\mathbf{Q}$.

Given two coarse spaces $(X, \mathcal{E})$ and $(X', \mathcal{E}')$, a mapping $f\colon X \longrightarrow X'$ is called *macro-uniform* if, for any $E \in \mathcal{E}$, there exists $E' \in \mathcal{E}'$ such that $f(E[x]) \subseteq E'[f(x)]$ for each $x \in X$. If f is a bijection such that f and f^{-1} are macro-uniform then f is called an *asymorphism.*

Theorem 3. *Let X be an unbounded finitary space, $\mathcal{K}_X$ denotes the class of coarse spaces asymorphic to X and let Y be an arbitrary coarse space. Then $Y \in \mathbf{QSP}\ \mathcal{K}_X$.*

Proof. By [5, Theorem 2], the minimal variety of coarse spaces containing $\mathcal{K}_X$ coincides with the class of all coarse spaces. By [5, Theorem 1], $Y \in \mathbf{QSP}\ \mathcal{K}_X$. □

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