FINITARY APPROXIMATIONS OF COARSE STRUCTURES


A coarse structure \(\mathcal{E}\) on a set \(X\) is called finitary if, for each entourage \(E \in \mathcal{E}\), there exists a natural number \(n\) such that \(E[x] < n\) for each \(x \in X\). By a finitary approximation of a coarse structure \(\mathcal{E}'\), we mean any finitary coarse structure \(\mathcal{E}\) such that \(\mathcal{E} \subseteq \mathcal{E}'\). If \(\mathcal{E}'\) has a countable base and \(E[x]\) is finite for each \(x \in X\) then \(\mathcal{E}'\) has a cellular finitary approximation \(\mathcal{E}\) such that the relations of linkness on subsets of \((X, \mathcal{E}')\) and \((X, \mathcal{E})\) coincide. This answers Question 6 from [8]: the class of cellular coarse spaces is not stable under linkness. We define and apply the strongest finitary approximation of a coarse structure.

1. Introduction. Given a set \(X\), a family \(\mathcal{E}\) of subsets of \(X \times X\) is called a coarse structure on \(X\) if

- each \(E \in \mathcal{E}\) contains the diagonal \(\Delta_X\), \(\Delta_X = \{(x, x) : x \in X\}\);
- if \(E, E' \in \mathcal{E}\) then \(E \circ E' \in \mathcal{E}\) and \(E^{-1} \in \mathcal{E}\), where \(E \circ E' = \{(x, y) : \exists z((x, z) \in E, (z, y) \in E')\}\), \(E^{-1} = \{(y, x) : (x, y) \in E\}\);
- if \(E \in \mathcal{E}\) and \(\Delta_X \subseteq E' \subseteq E\) then \(E' \in \mathcal{E}\);
- \(\bigcup \mathcal{E} = X \times X\).

A subfamily \(\mathcal{E}' \subseteq \mathcal{E}\) is called a base for \(\mathcal{E}\) if, for every \(E \in \mathcal{E}\), there exists \(E' \in \mathcal{E}'\) such that \(E \subseteq E'\). For \(x \in X\), \(A \subseteq X\) and \(E \in \mathcal{E}\), we denote

\[
E[x] = \{y \in X : (x, y) \in E\}, \quad E[A] = \bigcup_{a \in A} E[a], \quad E_A[x] = E[x] \cap A
\]

and say that \(E[x]\) and \(E[A]\) are balls of radius \(E\) around \(x\) and \(A\).

The pair \((X, \mathcal{E})\) is called a coarse space [10] or a ballean [7], [9].

For a coarse space \((X, \mathcal{E})\), a subset \(B \subseteq X\) is called bounded if \(B \subseteq E[x]\) for some \(E \in \mathcal{E}\) and \(x \in X\). The family \(\mathcal{B}(X, \mathcal{E})\) of all bounded subsets of \((X, \mathcal{E})\) is called the bornology of \((X, \mathcal{E})\). We recall that a family \(\mathcal{B}\) of subsets of a set \(X\) is a bornology if \(\mathcal{B}\) is closed under taking subsets and finite unions, and \(\mathcal{B}\) contains all finite subsets of \(X\).

We say that \((X, \mathcal{E})\) is locally finite, if each ball \(E[x]\) is finite, equivalently, \(\mathcal{B}(X, \mathcal{E}) = [X]^{<\omega}\).

A coarse space \((X, \mathcal{E})\) is called finitary, if for each \(E \in \mathcal{E}\) there exists a natural number \(n\) such that \(|E[x]| < n\) for each \(x \in X\).
Let $G$ be a transitive group of permutations of a set $X$. We denote by $X_G$ the set $X$ endowed with the coarse structure with the base
\[ \{ \{(x, gx) : g \in F\} : F \in [G]^{<\omega}, \text{id} \in F \} . \]

By [4, Theorem 1], for every finitary coarse structure $(X, \mathcal{E})$, there exists a transitive group $G$ of permutations of $X$ such that $(X, \mathcal{E}) = X_G$. For more general results, see [6).

By a finitary approximation of a coarse structure $\mathcal{E}'$ on $X$ we mean an arbitrary finitary coarse structure $\mathcal{E}$ on $X$ such that $\mathcal{E} \subseteq \mathcal{E}'$. In particular, the discrete coarse structure $\mathcal{E}_0$ with the base $\{ \{(x, y) : x, y \in F\} \cup \Delta_X : F \in [X]^{<\omega}\}$ is a finitary approximation of any coarse structure on $X$.

Following [8], we say that two subsets $A, B$ of a coarse space $(X, \mathcal{E})$ are linked (write $(A, B) \in \lambda_{(X, \mathcal{E})}$) if either $A, B$ are bounded or there exists $E \in \mathcal{E}$ such that $E[A] \cap B$ is unbounded.

A class $\mathcal{K}$ of coarse space is called $\lambda$-stable if $(X, \mathcal{E}) \in \mathcal{K}$ and $\lambda_{(X, \mathcal{E})} = \lambda_{(X, \mathcal{E}')}$ imply $(X, \mathcal{E}') \in \mathcal{K}$.

We recall that a coarse space $(X, \mathcal{E})$ is cellular if $\mathcal{E}$ has a base consisting of equivalences of $X$, equivalently, $\text{asdim } (X, \mathcal{E}) = 0$, see [9, Theorem 3.1.3].

This note is to give the negative ZFC-answer to the following question (see [8, Question 6] and [3, Problem 2.11]): is the class of cellular coarse spaces $\lambda$-stable? The same answer but under some set-theoretical assumptions involving small cardinals was obtained by Taras Banakh, see [8, Remark 1] and [1, Corollary 5.9].

2. Results.

**Theorem 1.** Let $(X, \mathcal{E}')$ be an unbounded locally finite coarse space with a countable base of $\mathcal{E}'$. Then there exists a cellular finitary approximation $\mathcal{E}$ of $X$ such that $\lambda_{(X, \mathcal{E})} = \lambda_{(X, \mathcal{E}')}$.

**Proof.** We use the Zorn Lemma to choose a maximal by inclusion cellular finitary approximation $\mathcal{E}$ of $\mathcal{E}'$ such that $\mathcal{E}_0 \subseteq \mathcal{E} \subseteq \mathcal{E}'$, $\mathcal{E}_0$ has the base $\{ \{(x, y) : x, y \in F\} \cup \Delta_X : F \in [X]^{<\omega}\}$. Let $\mathcal{P}$ be a base of $\mathcal{E}$ consisting of equivalences, and let $\{E_n : n < \omega\}$ be an increasing symmetric base of $\mathcal{E}'$. Since $(X, \mathcal{E}')$ is locally finite, we have $\lambda_{(X, \mathcal{E})} \subseteq \lambda_{(X, \mathcal{E}')}$. To prove $\lambda_{(X, \mathcal{E})} \subseteq \lambda_{(X, \mathcal{E}')}$, we take any two unbounded subsets $A, B$ of $X$ such that $(A, B) \in \lambda_{(X, \mathcal{E}')}$. Since $\mathcal{E} \subseteq \mathcal{E}'$, $\mathcal{E}'[A] \cap B$ is unbounded for some $E' \in \mathcal{E}'$, $E'[A] \cap B$ is unbounded for some $E' \in \mathcal{E}'$, $E'[A] \cap B$ is unbounded for some $E' \in \mathcal{E}'$, $E'[A] \cap B$ is unbounded for some $E' \in \mathcal{E}'$.

We choose inductively two injective sequences $(a_n)_{n<\omega}$ in $A$ and $(b_n)_{n<\omega}$ in $B$ such that $(a_n, b_n) \in E'$ and
\[ E'_{n+1}[\{a_{n+1}, b_{n+1}\}] \cap E'_{n+1}[\{a_0, b_0, \ldots, a_n, b_n\}] = \emptyset. \tag{*} \]

We put $M = \{ (a_n, b_n), (b_n, a_n) : n < \omega \} \cup \Delta_X$ and denote by $\mathcal{M}$ the smallest coarse structure on $X$ such that $M \in \mathcal{M}$ and $\mathcal{E} \subseteq \mathcal{M}$. Since $A, B$ are linked in $\mathcal{M}$, it suffices to show that $\mathcal{E} = \mathcal{M}$. In turn, on the maximality of $\mathcal{E}$ and $\mathcal{M} \subseteq \mathcal{E}'$, it suffices to verify that $\mathcal{M}$ is cellular.

We take an arbitrary $P \in \mathcal{P}$, a natural number $m$, put $H = M \cup P$ and show that $H^m$ is contained in some equivalence from $\mathcal{M}$. We choose a natural number $n$ such that $P \subseteq E'_{n+1}
$. If $x \in X$ and $E'_{n+1}[x] \cap E'_{n+1}[\{a_0, b_0, \ldots, a_n, b_n\}] = \emptyset$ then by (*)& either $H^m[x] = P[x]$ or $H^m[x]$ is the union of two blocks of $P$ containing some $\{a_k, b_k\}$, $k > n$. We enlarge the obtained partial equivalence of $X$ by the joining the block.
and get the desired equivalence from $\mathcal{M}$ containing $H^m[x]$. 

**Remark 1.** For every natural number $n$, there exists a coarse structure $\mathcal{E}'$ satisfying Theorem 1 such that $\operatorname{asdim} (X, \mathcal{E}') = n$. Analogously, there exists $\mathcal{E}'$ such that $\operatorname{asdim} (X, \mathcal{E}') = \infty$. Theorem 1 shows that the corresponding classes of coarse spaces are not $\lambda$-stable.

For a coarse structure $\mathcal{E}$ on $X$, we denote by $\mathcal{E}_{\text{fin}}$ the strongest finitary coarse structure on $X$ such that $\mathcal{E}_{\text{fin}} \subseteq \mathcal{E}$. We take the family $\mathcal{F} \in \mathcal{E}$ such that $E = E^{-1}$ and there exists a natural number $n$ such that $|E[x]| < n$ for every $x \in X$. Then $\mathcal{F}$ is a base of $\mathcal{E}_{\text{fin}}$.

**Proposition 1.** If a coarse structure $\mathcal{E}$ on $X$ is locally finite then $\lambda_{(X, \mathcal{E})} = \lambda_{(X, \mathcal{E}_{\text{fin}})}$.

**Proof.** Since $\mathcal{E}$ is locally finite, we have $\lambda_{(X, \mathcal{E}_{\text{fin}})} \subseteq \lambda_{(X, \mathcal{E})}$. Let $A, B$ be an infinite subset of $X$ such that $(A, B) \in \lambda_{(X, \mathcal{E})}$. We take $E \in \mathcal{E}$ such that $E = E^{-1}$ and $E[A] \cap B$ is infinite. We construct inductively a countable subset $A'$ of $A$ and an injective mapping $f : A' \to B$ such that $(x, f(x)) \in E$ for each $x \in A'$. Then the entourage $\{(x, f(x)) : x \in A'\} \cup \Delta_X$ in $\mathcal{E}_{\text{fin}}$ shows that $A, B$ are linked in $(X, \mathcal{E}_{\text{fin}})$. 

**Theorem 2.** Let $(X, \mathcal{E})$ be a locally finite coarse space. Assume that there exists $E \in \mathcal{E}$ such that, for every natural number $n$, there exists $x \in X$ such that $|E[x]| > n$. Then the coarse space $(X, \mathcal{E}_{\text{fin}})$ is not cellular.

**Proof.** We choose a sequence $(x_n)_{n<\omega}$ in $X$ such that the subsets $\{E[x_n] : n < \omega\}$ are pairwise disjoint and $|E[x_n]| > n$. Let $|E[x_n]| = m_n$. We enumerate $E[x_n] = \{a_{n1}, a_{n2}, \ldots, a_{nm_n}\}$ and define a bijection $f$ of $X$ such that, for each $n$, $f(a_{n1}) = f(a_{n2}), \ldots, f(a_{nm_n}) = a_{n1}$, and $f(x) = x$ for each $x \in X \setminus \bigcup_{k<\omega} E[x_n]$. Then the entourage $\{(x, f(x)) : x \in X\} \cup \Delta_X$ in $\mathcal{E}_{\text{fin}}$ shows that $\mathcal{E}_{\text{fin}}$ is not cellular. Otherwise, there is a equivalence $P$ in $\mathcal{E}_{\text{fin}}$ such that each $E[x_n]$ is contained in some block of $P$ contradicting finitarity of $\mathcal{E}_{\text{fin}}$.

**Remark 2.** If we take cellular $(X, \mathcal{E})$ in Theorem 2 and apply Proposition 1 then also get the negative answer to Question 6 from [7]. In light of Corollary 1 from [2] we may conjecture that $\operatorname{asdim} (X, \mathcal{E}_{\text{fin}}) = \infty$.

Following [8], we say that two subsets $A, B$ of $(X, \mathcal{E})$ are close (write $(A, B) \in \delta_{(X, \mathcal{E})}$) if there exists $E \in \mathcal{E}$ such that $A \subseteq E[B], B \subseteq E[A]$.

**Remark 3.** We show that Proposition 1 does not hold with $\delta_{(X, \mathcal{E})} = \delta_{(X, \mathcal{E}_{\text{fin}})}$ in place $\lambda_{(X, \mathcal{E})} = \lambda_{(X, \mathcal{E}_{\text{fin}})}$. We partition a countable set $X$ into finite subsets $\{X_n : n < \omega\}$ such that $|X_n| > n$, denote by $E$ the equivalence defined by this partition and take the smallest coarse structure $\mathcal{E}$ containing $E$ as an entourage. For each $n < \omega$, we pick $x_n \in X_n$, put $A = \{x_n : n < \omega\}$, $B = X$. Then $(A, B) \in \delta_{(X, \mathcal{E})}$ but $(A, B) \notin \delta_{(X, \mathcal{E}_{\text{fin}})}$.

By [1, Theorem 2.8], the strongest finitary coarse structure $\mathcal{F}$ on a set $X$ does not admit a cellular finitary approximation $\mathcal{E}$ such that $\delta_{(X, \mathcal{F})} = \delta_{(X, \mathcal{E})}$.

**Question.** Does there exist a non-cellular finitary coarse space $(X, \mathcal{E})$ such that $\mathcal{E}$ has a countable base and $\delta_{(X, \mathcal{E})} = \delta_{(X, \mathcal{E'})}$ for some cellular finitary approximation $\mathcal{E'}$ of $\mathcal{E}$?

Following [5], we say that a class $\mathcal{K}$ of coarse spaces is a variety if $\mathcal{K}$ is closed under operations of taking subspaces $\mathcal{S}$, cartesian products $\mathcal{P}$ and macro-uniform images $\mathcal{Q}$.
Given two coarse spaces \((X, \mathcal{E})\) and \((X', \mathcal{E}')\), a mapping \(f: X \to X'\) is called macro-uniform if, for any \(E \in \mathcal{E}\), there exists \(E' \in \mathcal{E}'\) such that \(f(E[x]) \subseteq E'[f(x)]\) for each \(x \in X\). If \(f\) is a bijection such that \(f\) and \(f^{-1}\) are macro-uniform then \(f\) is called an asymorphism.

**Theorem 3.** Let \(X\) be an unbounded finitary space, \(K_X\) denotes the class of coarse spaces asymorphic to \(X\) and let \(Y\) be an arbitrary coarse space. Then \(Y \in \text{QSP } K_X\).

**Proof.** By [5, Theorem 2], the minimal variety of coarse spaces containing \(K_X\) coincides with the class of all coarse spaces. By [5, Theorem 1], \(Y \in \text{QSP } K_X\).

**REFERENCES**

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