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B. V. Zabavsky, O. V. Domsha, O. M. Romaniv

CLEAR RINGS AND CLEAR ELEMENTS

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An element of a ring R is called clear if it is a sum of a unit-regular element and a unit. An associative ring is clear if each of its elements is clear. In this paper we define clear rings and extend many results to a wider class. Finally, we prove that a commutative Bézout domain is an elementary divisor ring if and only if every full 2×2 matrix over it is nontrivially clear.

1. Introduction. The results in this paper are prompted by looking at two sets $U(R)$ and $U_{reg}(R)$ in a ring R, which denote, respectively the unit group and the set of unit-regular elements in R. Certainly, the units and the unit-regular elements are key elements which determine a structure of the ring.

The study of rings generated additively by their units was started in 1953 when Wolfson [19] and Zelinsky [23] proved independently that every linear transformation of a vector space V over a division ring D is a sum of two nonsingular linear transformations except $\dim V = 1$ and $D = \mathbb{Z}_2$. It implies that every element of the ring of linear transformations $\mathrm{End}_D(V)$ is the sum of two units except one obvious case, when V is a one-dimensional space over $\mathbb{Z}_2$.

A ring in which every element is a sum of two units was called 2-good ring in the paper of Vamos [16]. The ring R is called von Neumann regular if for any $a \in R$ there exists $x \in R$ such that $axa = a$. In 1958 Skornyakov [15, Problem 20, p.167] formulated the question: «Is every element of von Neumann regular ring (which does not have $\mathbb{Z}_2$ as a factor-ring) a sum of units?»

According to Ehrlich [5], an element $a \in R$ is *unit-regular* if $a = aua$ for some unit $u \in U(R)$. It is easy to see that the element a is unit-regular if and only if the element a is a product of an idempotent and a unit. A ring is called *unit-regular* if every element is unit-regular.

Note that if $a \in R$ is a unit-regular element and 2 is a unit element in R then a can be written as $a = eu$, where e is an idempotent and u is a unit in R. Now, since 2 is a unit in R, $1 + e$ is a unit with $(1 + e)^{-1} = 1 - 2^{-1}e$. This gives that $e = (e + 1) - 1$ is a sum of two units, and hence a is a sum of two units.

We say that a ring R is *Henriksen elementary divisor ring* if for a square matrix $A \in R^{n\times n}$ there exist invertible matrices $P, Q \in R^{n\times n}$ such that PAQ is a diagonal matrix [20, p.10].

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Theorem 1 ([8], Theorem 11)**.** *Let R be a Henriksen elementary divisor ring. Then $R^{n\times n}$, $n>1$, is 2-good ring.*

It is important to note that every unit-regular ring is a Henriksen elementary divisor ring [7]. We also note that the units and idempotents are unit-regular elements.

The next class of rings generated additively by their units and idempotents is a class of clean ring. The notion of a clean ring was introduced in 1977 by Nicholson in [13]. Thereafter, such rings and their variations were intensively studied by many authors. Recall that an element of a ring R is *clean* if it is a sum of an idempotent and a unit of R. A ring R is *clean* if every element of R is clean [13]. Nicholson also showed that any unit-regular element is clean [13, Proposition 1.8] in a ring where idempotents are central. But a unit-regular element is not necessarily clean in a non-commutative ring. For example, the unit-regular matrix $\left(\begin{smallmatrix} 12 & 5 \\ 0 & 0 \end{smallmatrix}\right) \in \mathbb{Z}^{2\times 2}$ is not clean [9, Example 3.12]. At the same time, we note that Camillo and Yu showed that any unit-regular ring is clean [2, Theorem 5].

All commutative von Neumann regular rings, local rings, semi-perfect rings and the ring $R^{n\times n}$ for any clean ring R are examples of clean rings. The clean rings are closely connected to some important notions of the ring theory. Such rings are of interest since they constitute a subclass of the so-called exchange rings in the theory of non-commutative rings.

Module M_R has the exchange property if for every module A_R and any two decompositions

$$A = M' \bigoplus N = \bigoplus_{i\in I} A_i$$

with $M' \cong M$, there exist submodules $A_i' \subset A_i$ such that

$$A = M' \bigoplus \Big(\bigoplus_{i\in I} A_i\Big).$$

The module M_R has finite exchange property if the above condition is satisfied and the index set I is finite. Warfield [18] called a ring R an *exchange ring* if R_R has the finite exchange property, and he showed that this definition is left-right symmetric. Independently, Goodearl, Warfield [6] and Nicholson [13] obtained a very useful characterization of R: it is an exchange ring if and only if for any $a \in R$ there exists idempotent $e \in R$ such that $e \in aR$ and $1-e \in (1-a)R$. In this case the element a is called an exchange element.

In this paper we propose a concept of a clear ring based on the concept of a clear element. These results are mainly centered on the application to a classical and ancient problem of description of all elementary divisor rings. An overview can be found in [20]. In the case of commutative rings in [22] a connection of elementary divisor ring with the existence of clean elements of these rings is proved. Therefore, the problem of studying the matrix rings over elementary divisor rings in this aspect is significant.

We will study matrix rings over elementary divisor ring and reveal a connection to the theory of full matrices over certain classes of rings.

2. Notations and preliminary results. Let us introduce some notations and recall some definitions. Throughout the paper we suppose that R is an associative ring with non-zero unit and $U(R)$ is its group of units. The vector space of matrices over the ring R of size $k\times l$ is denoted by $R^{k\times l}$ and the group of units of the ring $R^{n\times n}$ by $GL_n(R)$. The Jacobson radical of R is denoted by $J(R)$. If $J(R)=0$, we say that R is a semi-simple ring.

A ring R is called a *right* (*left*) *Bézout ring* if each finitely generated right (left) ideal of R is principal. A ring R which is simultaneously right and left Bézout ring is called a *Bézout ring.*

A matrix A over the ring R *admits diagonal reduction* if there exist invertible matrices P and Q such that PAQ is a diagonal matrix (d_{ij}) for which d_{ii} is a total divisor $d_{i+1,i+1}$ (i.e. $Rd_{i+1,i+1}R \subseteq d_{ii}R \cap Rd_{ii}$) for each i. A ring R is called *an elementary divisor ring* provided that every matrix over R admits a diagonal reduction [20].

We can define ranks of a matrix A over R on their rows $\rho_r(A)$ and their columns $\rho_c(A)$, respectively (see [3, p. 247]). The smallest $m \in \mathbb{N}$ such that the matrix $A \in R^{k\times l}$ is a product of two matrices of size $k \times m$ and $m \times l$, is called an *inner rank* $\rho(A)$ of A. Note that $\rho(A) \leq \min\{\rho_r(A), \rho_c(A)\}$ and elementary transformations do not change the number $\rho(A)$. If R is a right Bézout domain then $\rho(A) = \rho_r(A) = \rho_c(A)$ for any A over R. A matrix $A \in R^{n\times n}$ is called *full* if $\rho(A) = n$ (see [3, p. 248]), i.e. $R^{n\times n}AR^{n\times n} = R^{n\times n}$.

In the sequel, we use the following result.

Proposition 1. *The following statements hold:*

i) ([10, Corollary 3.7]) *Let $R = S^{2\times 2}$, where S is any ring and $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in R$. If $b \in U(S)$ or $c \in U(S)$ then A is a clean element of $S^{2\times 2}$.*

ii) ([4, Theorem 2]) *A commutative Bézout domain R is an elementary divisor ring if and only if for any nonsingular full matrix $A \in R^{2\times 2}$ a right (left) principal ideal $AR^{2\times 2}$ ($R^{2\times 2}A$) contains nontrivial idempotent.*

iii) ([13, Proposition 1.8]) *Every clean element of a ring is an exchange element.*

3. Clear elements and clear rings. Different classes of rings have generalized clean rings by adding one or more adjectives such as «almost», «semi», «uniquely», «G», «unit» etc. to «clean» (for example, [1, 11, 24]). We propose another generalization of clean elements, namely clear elements. The description of full matrices over commutative elementary divisor rings based on this concept is possible.

An element a of a ring R is *clear* if $a = r + u$, where r is a unit-regular element and $u \in U(R)$. A ring R is *clear* if every element is clear. The obvious examples of clear elements are 0 and 1.

Let $a \in R$ be a clear element such that $a = r + u$, where r is unit-regular and $u \in U(R)$. If $r \neq 0$ and $r \notin U(R)$, we say that the clear element a is nontrivial. For example, if R is a 2-good ring (any element is the sum of two units) then any $a \in R$ has trivial representation $a = u + v$, where $u, v \in U(R)$, though it does not exclude a nontrivial representation of elements of R.

Since an idempotent of a ring is obviously a unit-regular element, we get the following result.

Proposition 2. *Any clean element is clear. A clean ring is clear.*

Moreover, we have the following proposition.

Proposition 3. *Any unit-regular element is clear.*

Proof. Let $aua = a$ for some $u \in U(R)$. Since au is idempotent, so $au = 1 - e$ for some idempotent $e \in R$. Then $a = u^{-1} - eu^{-1}$. Since $-eu^{-1}$ is unit-regular, then a is a clear element. □

We point out that the converse statements of these propositions are not valid. For example, the matrix $\left(\begin{smallmatrix} 12 & 5 \\ 0 & 0 \end{smallmatrix}\right) \in \mathbb{Z}^{2\times 2}$ is a clear element, but as noted earlier, it is not clean. The ring $\mathbb{Z}_4$ is a clear ring, but it is not unit-regular.

The description of idempotent or unit-regular matrices in the rings of matrices over rings is rather an actual task. For example:

Proposition 4 ([14], Proposition 2.1)**.** *Let $R = S^{2\times 2}$, where S is any ring and let $A = \left(\begin{smallmatrix} a & 0 \\ b & 0 \end{smallmatrix}\right) \in R$. A matrix A is unit-regular in R if and only if there exists an idempotent $e \in S$ and a unimodular column $\left(\begin{smallmatrix} a_1 \\ b_1 \end{smallmatrix}\right) \in S^{1\times 2}$ such that $\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) = \left(\begin{smallmatrix} a_1 \\ b_1 \end{smallmatrix}\right) e$.*

Recall that a column $\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)$ is unimodular if $Ra + Rb = R$. Also we note that this condition is obviously equivalent to $Ra + Rb = Re$, where $e^2 = e$, $a = a_1 e$, $b = b_1 e$ and $Ra_1 + Rb_1 = R$.

Remind that every unit-regular element is a product of an idempotent and a unit element and the multiplication is preserved by every ring homomorphism. Therefore, the homomorphic images of an idempotent and a unit element are idempotent and unit elements of the corresponding ring respectively.

Proposition 5. *Every homorphic image of a clear ring is clear.*

It is known that the multiplication in direct product of rings is defined component-wise, so an element in the direct product of rings is unit (resp. idempotent) of that ring if and only if the element in each of its components is unit (resp. idempotent) of this ring. Since the addition in a direct product of rings is also defined component-wise, the result follows from a simple computation.

Proposition 6. *Every direct product of clear rings is clear.*

Lemma 1. ([1]) *An element $a \in R$ is clear if and only if ua and au are clean elements for some $u \in U(R)$.*

Proof. Let a be a clear element, i.e. $a = r + v$, where r is a unit-regular element and $v \in U(R)$. Since $rur = r$ for some $u \in U(R)$, we have $ua = ur + uv$, where $(ur)^2 = ur$ and $uv \in U(R)$, i.e. ua is clean element. Similarly, $au = ru + vu$, where $(ru)^2 = ru$ and $vu \in U(R)$.

Let for $a \in R$ there exists unit element $u \in U(R)$ such that ua and au are clean elements. Let $ua = e + v$, where $e^2 = e$ and $v \in U(R)$. Then $a = u^{-1}e + u^{-1}v$. Since $(u^{-1}e)u(u^{-1}e) = u^{-1}e$, we have that $u^{-1}e$ is an unit-regular element and $u^{-1}v \in U(R)$, i.e. a is clear. The same is in the case au. □

In this context for a description of clear rings let us give the following definition.

A ring R is said to have unit-regular stable range 1 if for any element $a, b \in R$ with $aR + bR = R$ there exists some unit-regular element r of R such that $a + br$ is a unit element of R.

Lemma 2. *A ring of unit-regular stable range 1 is clear.*

Proof. Let R be a ring of unit-regular stable range 1 and $a \in R$. Then $aR + (-1)R = R$ and we have that $a + (-1)r = u \in U(R)$ for some unit-regular element $r \in R$, i.e. $a = r + u$. □

For the following results we recall that an element $a \in R$ is *2-clean* if $a = e + u + v$ for some idempotent e and unit elements $u, v \in U(R)$ [12].

Proposition 7. *For any commutative ring any clear element is 2-clean. Any clear ring is 2-clean ring.*

Proof. Let $a \in R$ be a clear element, i.e. $a = r + u$ for some unit-regular element r and $u \in U(R)$. Since every unit-regular element of a commutative ring is clean [17] we have that $r = e + v$ for some idempotent $e \in R$ and $v \in U(R)$. Then $a = e + u + v$. □

In addition, we shall notice that over any ring 2×2 and 3×3 matrices are 2-clean [17, Lemma 3].

Since any local ring is clean, by Proposition 2 we also have the following result.

Proposition 8. *Any local ring is clear.*

There are not nontrivial idempotents in the local ring. Let us describe clear rings R which have not nontrivial idempotents (domain, for example). Recall that an element $a \in R$ is *2-good* if a is a sum of two unit elements.

Proposition 9. *The following statements are equivalent for any ring R:*

1) *R is a clear and has no nontrivial idempotents;*
2) *for any element $a \in R$: a) $a \in U(R)$ or b) a is 2-good.*

Proof. Let R be clear and have not nontrivial idempotents.

Since 0 and 1 are the only idempotents in R, 0 and $u \in U(R)$ are unit-regulars, i.e. all elements of R are trivial clear elements, that is for any $a \in R$ we have that $a \in U(R)$ or a is a 2-good element.

The implication (2) $\Longrightarrow$ (1) is obvious. □

Lemma 3. *Let R be a commutative elementary divisor ring. Then for any full matrix $A \in R^{2\times 2}$ there exist invertible matrices $P, Q \in GL_2(R)$ such that $PAQ = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}$ for some element $d \in R$.*

Proof. We noted earlier that if R is a commutative elementary divisor ring then for any full matrix $A \in R^{2\times 2}$ there exist invertible matrices $P, T \in GL_2(R)$ such that

$$PAT = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} = D,$$

where d_1 is a total divisor of d_2. Since R is a commutative ring, obviously d_1 is a divisor of d_2. From

$$R = R^{2\times 2} A\, R^{2\times 2} = R^{2\times 2} D R^{2\times 2}$$

and from d_1 being a divisor of d_2 follows $d_1 \in U(R)$, i.e. we can assume that $d_1 = 1$. □

Theorem 2. *Let R be a commutative elementary divisor ring and A be a full nonsingular matrix of $R^{2\times 2}$. Then exist invertible matrices $P, Q \in GL_2(R)$ such that PAQ is nontrivial clear element of $R^{2\times 2}$.*

Proof. We note that if $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & d \end{smallmatrix}\right) \in S^{2\times 2}$ for any ring S then

$$\begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ d & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ d & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ d+1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

So (by Proposition 1), we have that $\left(\begin{smallmatrix} 0 & 1 \\ d & 0 \end{smallmatrix}\right)$ is a clean matrix. Also we note that $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \in GL_2(R)$.

According to our comments and Lemma 3 we finish the proof of Theorem 2. □

Theorem 3. *Let R be a commutative elementary divisor ring. Then every full nonsingular matrix $A \in R^{2\times 2}$ is nontrivially clear.*

Proof. By Theorem 2 we have that

$$PAQ = E + U,$$

where $E^2 = E$ and invertible matrices $P, Q, U \in GL_2(R)$. So $A = P^{-1}EQ^{-1} + P^{-1}UQ^{-1}$ and $P^{-1}EQ^{-1} = (P^{-1}EQ^{-1})QP(P^{-1}EQ^{-1})$. Obviously, $QP \in GL_2(R)$ and $P^{-1}EQ^{-1}$ is a unit-regular matrix and $P^{-1}UQ^{-1} \in GL_2(R)$. Thus, A is clear matrix. □

It is well known that every commutative principal ideal domain is an elementary divisor domain [14]. As a consequence of Theorem 3 we have the following result.

Corollary 1. *Let R be a commutative principal ideal domain. Then any full matrix $A \in R^{2\times 2}$ is clear.*

Now let R be a commutative Bézout domain in which any nonzero prime ideal is contained in a unique maximal ideal, i.e. R is a PM^* ring. By [21, Theorem 1], R is an elementary divisor ring. So, in the same way from Theorem 3 we get the following result.

Corollary 2. *Let R be a commutative PM^* Bézout domain. Then any full matrix $A \in R^{2\times 2}$ is clear.*

Note that an elementary divisor ring is a Bézout ring [20, Theorem 1.2.7]. As a consequence of Theorem 3 and Proposition 1(ii) we can prove Theorem 4.

Theorem 4. *Let R be a semi-simple commutative Bézout domain. The following statements are equivalent:*

1) *R is an elementary divisor ring;*
2) *any full nonsingular matrix of $R^{2\times 2}$ is nontrivially clear.*

Proof. By Theorem 3 we have the implication 1) $\Longrightarrow$ 2).

2) $\Longrightarrow$ 1) Let a full matrix $A \in R^{2\times 2}$ is be clear. By Lemma 1 there exists an invertible matrix $U \in GL_2(R)$ such that UA and AU are clean matrices. By Proposition 1 UA (AU) is an exchange element. By [13, Proposition 1.1] a right (left) ideal $AUR^{2\times 2}$ $(R^{2\times 2}UA)$ contains an idempotent E (F). Since R is semi-simple and $J(R^{2\times 2}) = (J(R))^{2\times 2}$ by proof of Proposition 1.9 [13] the idempotent E (F) is not trivial. By Proposition 1(ii) we obtain that R is an elementary divisor domain. □

The following theorem establishes the place of the main results of this paper in the general system of interrelations of the results obtained by the authors.

Theorem 5. *Let R be a commutative Bezout domain. Ring $R^{2\times 2}$ is a clear if and only if $R^{2\times 2}$ is 2-good ring.*

Proof. Note that if $R^{2\times 2}$ is 2-good ring, obviously $R^{2\times 2}$ is a clear ring. If $R^{2\times 2}$ is a clear ring then for any matrix $A \in R^{2\times 2}$ we have that A is 2-good or $AR^{2\times 2}$ contains a nontrivial idempotent. For details, see example [1, Proposition 5.10]. If $AR^{2\times 2}$ contains a nontrivial idempotent for any matrix $A \in R^{2\times 2}$, by Theorem 4 we have that R is an elementary divisor ring. By Theorem 1 we have that $R^{2\times 2}$ is a 2-good ring. □

4. Some open questions.

1. *Is a commutative clear ring a ring of unit-regular stable range 1?*

2. *Is the notion of a ring of unit-regular stable range 1 left-right symmetric?*

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Department of Algebra and Logic
Ivan Franko National University of Lviv
Lviv, Ukraine
zabavskii@gmail.com

Lviv Regional Institute for Public Administration
of the National Academy for Public Administration
under the President of Ukraine
Lviv, Ukraine
olya.domsha@ukr.net

Department of Algebra and Logic, Ivan Franko National University of Lviv
Lviv, Ukraine
oleh.romaniv@lnu.edu.ua

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