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ON THE IDEMPOTENT AND NILPOTENT SUM NUMBERS OF MATRICES OVER CERTAIN INDECOMPOSABLE RINGS AND RELATED CONCEPTS

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We investigate a few special decompositions in arbitrary rings and matrix rings over indecomposable rings into nilpotent and idempotent elements. Moreover, we also define and study the nilpotent sum trace number of nilpotent matrices over an arbitrary ring. Some related notions are explored as well.

The paper is dedicated to the memory of Professor Bohdan Zabavsky

1. Introduction and definitions. Everywhere in the text of the present paper all our rings, usually denoted by R, are assumed to be associative, containing the identity element 1 which differs from the zero element 0. Our terminology and notations are at most part standard and mainly in agreement with [23]. For instance, for such a ring R, $J(R)$ denotes the Jacobson radical. In addition, the letter $\mathbb{M}_n(R)$ stands for the full $n \times n$ matrix ring over a ring R whenever $n \in \mathbb{N}$. Also, $\mathbb{Z}_n \cong \mathbb{Z}/(n)$ will mean the ring of all integers modulo the principal ideal (n) generated by a fixed positive integer (= a natural number) n. We will be mainly focussed on matrices taken over the indecomposable ring $\mathbb{Z}_4 = \{0, 1, 2, 3 \mid 4 = 0\}$, consisting of precisely four elements.

Likewise, the more specific notions will be provided in what follows. An element r of a ring R is called *periodic* if there exist two different positive integers m, n both depending on r such that $r^m = r^n$. In particular, if $n = 1$, r is said to be *potent*, that is, $r^m = r$. In addition, when $m = 2$, r is called an *idempotent*, i.e., $r^2 = r$ and, when $m = 3$, r is called a *tripotent*, i.e., $r^3 = r$.

As the title of the article unambiguously illustrates, both the idempotent and nilpotent elements will play a key role in our further explorations. Certain principally known results concerning these two types of elements associating them with some special decompositions in ordinary rings and matrix rings are as follows (we shall list only the most important of them which definitely affect on the present work): In [1] were considered those matrices which are sums of nilpotent ones. Also, in [2] and [28] respectively were obtained some decompositions of matrices over the field $\mathbb{Z}_2$ of two elements. Moreover, in [29] were established some matrix presentations over the indecomposable ring $\mathbb{Z}_4$ and some other commutative rings (resp.,

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fields) into the sum of three idempotents or three involutions. Likewise, in [25, 26] were considered those (infinite) matrices which are expressed as sums of three or more idempotents (see [27] too).

On the other side, in [22] were characterized those rings whose elements are sums of two (commuting) idempotents. It is well to note that these results were substantially improved in [5]–[13], respectively.

Motivated entirely by the already described achievements, we are in a position to initiate in what follows an examination in a slightly different way. And so, we come to the following new concept which states as follows:

Definition 1. Suppose R is a ring. Then we shall write $\mathrm{sntr}(\mathbb{M}_n(R)) = m$ if m is the minimal natural number such that the trace of each nilpotent matrix over R is a sum of m nilpotents. If such an integer m does not exist, but the trace of every nilpotent matrix is a sum of nilpotents, we shall write $\mathrm{sntr}(\mathbb{M}_n(R)) = \omega$. If, however, there exists a nilpotent matrix over R such that its trace is *not* a sum of nilpotents, then we shall just write $\mathrm{sntr}(\mathbb{M}_n(R)) = \infty$.

Computing the so-defined number will shed some more light on the rather complicated structure of the nilpotent matrices and their presentations.

On the other vein, in [19] was introduced the so-named "*idempotent sum number*", denoted by "**isn**", which is, merely, in a sharp transversal with the above definition by considering a minimal sum of idempotents.

The goal that we pursue here is to make up this article to be the frontier in this fascinating subject by introducing the reader in all the methods used for computation of "**sntr**" and "**isn**" for some matrix rings over concrete rings. Some other related matrix questions that could be of some continuing interest and importance are also considered in detail. Our results are selected in the subsequent section which contains exactly three subsections containing a different material.

2. Sums of idempotents and nilpotents in (matrix) rings. We distribute our work into three subsections like these:

2.1. Examples and some results. We start here with a well-known statement. In fact, it is well known that the trace of a nilpotent matrix over an arbitrary commutative ring is always a nilpotent (and so it must be zero over a field).

Example 1. If K is a commutative ring (in particular, if K is a field), then $\mathrm{sntr}(\mathbb{M}_n(K)) = 1$.

However, it is not the case that the trace of a nilpotent matrix over a non-commutative ring is also nilpotent. Specifically, the following holds:

Example 2. There is a nilpotent matrix defined over a non-commutative ring such that its trace is not nilpotent.

Proof. Let R be the quotient of the free non-commutative ring on two variables x, y by the ideal generated by x^2 and y^2; so $R = \mathbb{Z}\langle x, y\rangle/(x^2, y^2)$. Suppose now that $A = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}$ is the 2×2 matrix over R, where $X = x+(x^2, y^2)$ and $Y = y+(x^2, y^2)$ denote the images of x and y in R, respectively. Then $A^2 = 0$, and thus A is a nilpotent. But the trace $X+Y$ of A is not a nilpotent, since any power of $X+Y$ is of the form $(XY)^m+(YX)^m$ or $Y(XY)^m+X(YX)^m$, once all terms containing X^2 or Y^2 are identified with 0. But expressions of these forms are not 0 in R, because expressions of the forms $(xy)^m + (yx)^m$ and $y(xy)^m + x(yx)^m$ are not elements of the ideal (x^2, y^2) in the ring $\mathbb{Z}\langle x, y\rangle$. $\square$

In regard to the preceding example, the next construction is somewhat surprising.

Example 3. If D is a non-commutative division ring, then $\mathrm{sntr}(\mathbb{M}_n(D)) = \infty$.

Proof. Given $x, y \in D$ which not commute, then one checks that the matrix $\begin{pmatrix} -xy & x \\ -yxy & yx \end{pmatrix}$ is a nilpotent of order 2, but its trace $-xy + yx$ is obviously not zero, however. Therefore, it is definitely not a sum of nilpotent elements since 0 is the only one nilpotent element in D. $\square$

Notice that due to the classical Wedderburn's theorem, the division ring D from the previous example has to be infinite (see also [20]).

The last example may also be subsumed by the following construction: Let S be a unital ring, let T be the ring of 2×2 matrices over S, and let R be the ring of 2×2 matrices over T. In T, let $E_{11}, E_{12}, E_{21}, E_{22}$ denote the matrix units (i.e., matrices with one coordinate 1, and the other coordinates 0. So, for instance, E_{11} has a 1 in the first position on the first row, and 0s elsewhere). Now, consider the following matrix $M = \begin{pmatrix} E_{11} & -E_{12} \\ E_{21} & -E_{22} \end{pmatrix}$ in R.

Then one verifies that $M^2 = 0$, and the trace of M is the matrix $E_{11} - E_{22}$ in T which is, in particular, not nilpotent. Now, depending on what S is, $E_{11} - E_{22}$ could be the sum of nilpotent elements in T; in this aspect, based on certain calculations which we leave to the interested reader, that seems to be the case provided S is a field. However, it is almost sure that there are rings S for which it would not.

The following rather curious statement is well-known (see [21, Lemma 1]), but also somewhat appeared in [24] as well as is specially treated in [9] and [16], respectively. We, however, shall state the complete proof for fullness of the exposition and for the readers' convenience.

Lemma 1. *Any nilpotent matrix over a field is the difference of two idempotent matrices.*

Proof. Take a nilpotent matrix N over a field F. Standardly, put N in Jordan form, possible because all its eigenvalues (0) lie in F. The only property we really need now of N is that it is strictly upper triangular with all its nonzero entries in the first super-diagonal. In fact, even much more weaker than that – nonzero entries have odd-parity of positions (i, j), meaning if i is even, then j is odd, and the other way round. Let E be the diagonal idempotent matrix with diagonal entries the sequence $1, 0, 1, 0, 1, \cdots$ as follows

$$E = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

Observe at once that for a general matrix A, the matrix EAE retains the entries of A in the odd, odd positions and sets everything else to 0. Similarly, $(I - E)A(I - E)$ keeps the even, even elements and wipes out the rest, where I is the identity matrix. Hence, it is immediate that $ENE = (I - E)N(I - E) = 0$. Therefore,

$$N = ENE + EN(I - E) + (I - E)NE + (I - E)N(I - E) =$$
$$= EN(I - E) + (I - E)NE = [E + EN(I - E)] - [E - (I - E)NE],$$

which gives N as the difference of the two square-bracketed idempotents, as required. $\square$

It was proved in [25] and [26] that, for any $n \geq 1$, every matrix in $\mathbb{M}_n(\mathbb{Z}_2)$ is a sum of three idempotents. In accordance with [22] and the terminology from [19], one asserts that $\mathrm{isn}(\mathbb{M}_n(\mathbb{Z}_2)) = 3$. We are now ready to state a new more transparent and conceptual proof in a slightly more generalized form. Specifically, the following holds:

Theorem 1. *Each matrix A of $\mathbb{M}_n(\mathbb{Z}_2)$ is of the form $E_1 + E_2 + E_3$, where E_1, E_2, E_3 are idempotent matrices with the property that E_1E_2 and E_2E_1 are tripotents.*

Proof. Consulting with [2, Theorem 3], A can be represented as $A = N + F$, where N is a nilpotent matrix and F is an idempotent matrix. Applying now Lemma 1, N is the sum of two idempotent matrices, say $E_1 + E_2$, as the characteristic is exactly 2, and we are done.

Letting now we have the record $A = E_1 + E_2 + E_3$ with E_1, E_2, E_3 idempotents (for convenience, we just replaced F by E_3). In virtue of [28, Theorem 2.2], one may write that $(E_1 + E_2)^4 = 0$. Taking into account that the relation $2 = 0$ is still fulfilled here, one deduces that

$$E_1 + E_2 + E_1E_2 + E_2E_1 + E_1E_2E_1 + E_2E_1E_2 + (E_1E_2)^2 + (E_2E_1)^2 = 0.$$

Multiplying both sides of the last equality by E_1, one finds that $E_1 = (E_1E_2)^2E_1$ allowing us to conclude that $E_1E_2 = (E_1E_2)^3$, as stated. Analogously, repeating the same procedure for E_2, one derives that $(E_2E_1)^3 = E_2E_1$ and the proof is over. $\square$

In contrast to [29, Example 3.5] the following is true:

Example 4. The matrices $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ are sums of four idempotent matrices over $\mathbb{Z}_4$.

Proof. One checks directly that $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$,
and that $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$. $\square$

These two presentations allow us to proceed by proving the following.

Proposition 1. *The equality* $\mathrm{isn}(\mathbb{M}_2(\mathbb{Z}_4)) = 4$ *is valid.*

Proof. We shall detect foremost that all unipotents (i.e., the sums of 1 and a nilpotent) in $\mathbb{M}_2(\mathbb{Z}_4)$ are sums of four idempotents. In fact, this follows by direct hand-written calculations and, that is why, we will omit the details leaving them to the interested reader for an inspection. Next, the proof goes in a standard way, and so we are thought. $\square$

The next constructions show that such an equality is not longer true for rings of the type $\mathbb{Z}_{2^k}$, where $k > 2$, namely there is a simple class of finite commutative local rings having the properties:

Example 5. There exist finite commutative local rings for which there is no bound for the number of idempotents needed for expressing 2×2 matrices as sums of idempotents.

Proof. Consider the ring $S = \mathbb{Z}_{2^m}$ for $m \geq 3$. Then, it is plainly checked that all proper idempotents in $\mathbb{M}_2(S)$ have trace 1. Let $N = \begin{pmatrix} 0 & 0 \\ 0 & 2^{m-1} \end{pmatrix}$.

Then, one verifies that N is nilpotent and the unipotent $U = I + N$ has trace $t = 2 + 2^{m-1}$. Therefore, U cannot be the sum of fewer than t proper idempotents. E.g., for $S = \mathbb{Z}_{16}$ and

$N = \begin{pmatrix} 0 & 0 \\ 0 & 8 \end{pmatrix}$, the unipotent $I + N$ is *not* the sum of fewer than 10 proper idempotents and *not* the sum of 6 idempotents, allowing for the idempotent I (of trace 2). □

Our next result, which is closely related to the expression of matrices as sums of idempotent ones, is the following:

Proposition 2. *For an arbitrary ring R and an integer $n > 1$ every element of $\mathbb{M}_n(R)$ can be expressed as a finite sum of elements, each of which is either an idempotent or a product of two idempotents.*

Proof. We consider first the case where $n = 2$. To that purpose, let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and set $M_1 = \begin{pmatrix} 0 & b \\ 0 & 1 \end{pmatrix}$, $M_2 = \begin{pmatrix} 0 & 0 \\ c & 1 \end{pmatrix}$, $M_3 = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ and $M_4 = \begin{pmatrix} 0 & 0 \\ 0 & d-2 \end{pmatrix}$. Clearly, $M = M_1 + M_2 + M_3 + M_4$. A straightforward check shows that M_1, M_2 are both idempotents.

However, one verifies that $M_3 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ a-1 & 0 \end{pmatrix} = XY$ say, and it is easily checked that both X, Y are idempotents.

Finally, $M_4 = \begin{pmatrix} 0 & 0 \\ d-3 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = ZW$ say, and again it is easy to verify that both Z, W are idempotents. So, the result is true for $n = 2$.

Proceeding by induction, assume the result holds for $n = k$ and consider $M = \begin{pmatrix} A & b \\ c & d \end{pmatrix}$, a $(k+1) \times (k+1)$ matrix, where A is a $k \times k$ matrix, b is a $k \times 1$ column vector, c is a $1 \times k$ row vector and $d \in R$. Let O denote the $k \times k$ zero matrix. then

$$M = \begin{pmatrix} O & b \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} O & 0 \\ c & 1 \end{pmatrix} + \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} O & 0 \\ 0 & d-2 \end{pmatrix}.$$

Notice that the first two matrices in this sum are readily seen to be idempotents, and an identical argument to that used in the 2×2 case shows that the final matrix is a product of two idempotents.

What needs to establish is that the third matrix $\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$ is a sum of idempotents and products of two idempotents. Indeed, by the induction hypothesis, the matrix A may be expressed as $A = A_1 + \cdots + A_t$ for some finite index t, where each A_i is either an idempotent or a product of two idempotents $(1 \leq i \leq t)$. However, if X is an idempotent $k \times k$ matrix, then $\begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}$ is an idempotent $(k+1) \times (k+1)$ matrix, while if X, Y are idempotents, then $\begin{pmatrix} XY & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} Y & 0 \\ 0 & 0 \end{pmatrix}$ is a product of two $(k+1) \times (k+1)$ idempotent matrices. Thus the remaining matrix under description can be expressed in the wanted form. Consequently, the initial matrix M has the same property, as expected. □

We end this subsection with the following non-trivial result regarding another special matrix decomposition over $\mathbb{Z}_4$. For a further generalization in this directory, we refer to [18], where a more powerful machinery is used.

Theorem 2. *For all $n \geq 1$, each matrix in $\mathbb{M}_n(\mathbb{Z}_4)$ is decomposable as a sum of a nilpotent matrix of order less than or equal to 4 and a periodic matrix.*

Proof. It was proven in [17] that every matrix in $\mathbb{M}_n(\mathbb{Z}_2)$ is the sum of a square-zero nilpotent and a potent. We shall freely use this fact in the sequel. To that goal, set $P := \mathbb{M}_n(\mathbb{Z}_4)$. It is well known that $J(P) = J(\mathbb{M}_n(\mathbb{Z}_4)) = \mathbb{M}_n(J(\mathbb{Z}_4))$ – see, e.g., [23]. As $J(\mathbb{Z}_4) = \{0, 2\}$ with $2^2 = 0$, it readily follows that $J^2(P) = \{0\}$, that is, $J(P)$ is a nil-ideal of exponent exactly 2. Moreover, since $\mathbb{Z}_4/J(\mathbb{Z}_4) \cong \mathbb{Z}_2$, it is only a routine technical exercise to verify that $P/J(P) \cong \mathbb{M}_n(\mathbb{Z}_2)$. Henceforth, for any $x \in P$, one writes by what we have commented above that $x+J(P) = [q+J(P)]+[p+J(P)]$, where the first term is a nilpotent of exponent 2 and the second term is a potent. Consequently, $q^2 \in J(P)$ and $p^t - p \in J(P)$ for some natural t. The latter relation gives that $(p^t - p)^2 = 0$ whence $p^{2t} = 2p^{t+1} - p^2$. Squaring this, we just obtain that $p^{4t} = p^4$ because $4 = 0$ in P. Thus p is a potent element. Furthermore, one sees that $x = (q + j) + p$ for some $j \in J(P)$. But $(q + j)^2 \in J(P)$ and hence $(q + j)^4 = 0$, as required. □

2.2. Decompositions in rings. We will be now concerned with some special decompositions in arbitrary rings which could be eventually very useful in finding the linear expressions of matrices as sums and/or differences of idempotents — see also [5]-[8] as well as [10]-[13].

Before doing that, we need a few technicalities.

Lemma 2. *In a ring R whose elements are (either) a sum of three idempotents or a minus sum of two idempotents, the relation $2^2.3^4.5 = 0$ holds.*

Proof. First, write $-3 = -e - f$ for some two idempotents e, f from the ring. Hence, one checks that $ef = fe$. Since $1 - e = -2 + f$, we deduce by squaring that $4f = 6$. Similarly, $4e = 6$ and so $16ef = 36$. Squaring now $3 = e + f$, it follows that $6 = 2ef$ whence $48 = 36$ which yields that $12 = 2^2.3 = 0$, as pursued.

Letting now $-3 = e+f+h$ for some three idempotents e, f, h from the ring, we have that $-3-e = f+h$. Squaring this equality, we infer that $12+8e = 12+8(-3-f-h) = fh+hf$ and thus $-12 - 8f - 8h = fh + hf$. Multiplying this by f from the left and from the right, respectively, we derive that $-20f - 9fh = fhf = -20f - 9hf$ and thus $9fh = 9hf$. Furthermore, the multiplication of $-20f-9fh = fhf$ by 9 gives that $-180f = 90hf$ and next the multiplication of the last by h from the left gives that $270hf = 270fh = 0$. Therefore, $12 + 8e = fh + hf$ ensures that $12.135 + 8.135e = 15.9fh + 15.9hf = 15.18fh = 270fh = 0$ as $9hf = 9fh$. That is why, $8.135e = -12.135$ and reasoning in the same way we get that $8.135e = 8.135f = 8.135h = -12.135$. Finally, one concludes that $-3.8.135 = (e + f + h).8.135$ yielding $-24.135 = -36.135$, so that $2^2.3^4.5 = 0$, as stated. □

Lemma 3. *In a ring R whose elements are (either) a sum of three idempotents or a difference of two idempotents, the relation $2^2.3.5 = 0$ holds.*

Proof. Firstly, write $4 = e_1 - e - 2$ for some two idempotents e_1, e_2 of the ring. Hence, it is easily observed that $e_1e_2 = e_2e_1$ and so $4^3 = 4$ as $(e_1-e_2)^3 = e-1-e_2$. Thus $60 = 4.3.5 = 0$, as asked for.

Secondly, let us write that $4 = e_1 + e_2 + e_3$ for some three idempotents e_1, e_2, e_3 of the ring. So, $4-e_1 = e_2+e_3$ and, by squaring, one detects that $12-6e_1 = e_2e_3+e_3e_2$. This riches us that $6e_2 + 6e_3 - 12 = e_2e_3 + e_3e_2$ and multiplying both sides of this equality by e_2 from the left and from the right, respectively, one infers that $5e_2e_3 - 6e_2 = e_2e_3e_2 = 5e_3e_2 - 6e_2$ giving up $5e_2e_3 = 5e_3e_2$. Therefore, $25e_2e_3 - 30e_2 = 5e_2e_3e_2 = 5e_3e_2 = 5e_2e_3$ forcing at once that $20e_2e_3 = 30e_2$, whence $20e_2e_3 = 30e_2e_3$ and $10e_2e_3 = 0$. That is why, we will have that $60 - 30e_1 = 5e_2e_3 + 5e_3e_2 = 10e_2e_3 = 0$ which means that $30e_1 = 60$ and also makes sense that $30e_1 = 60e_1$, i.e., $30e_1 = 0 = 60$, that is, $4.3.5 = 0$, as formulated. □

Lemma 4. *In a ring R whose elements are (either) a sum or a minus sum of three idempotents, the relation $2^3.3.5.7.11 = 0$ holds.*

Proof. Writing $4 = e + f + h$ for some three idempotents e, f, h in the ring. Thus $1 - e = -3 + f + h$. By multiplying that equality on the left and on the right by e, one derives that $ef + eh = fe + he = 3e$. Moreover, squaring the equality $4 = e + f + h$, one deduces that $12 = 6e + fh + hf = 6(4 - f - h) + fh + hf$, i.e., that $12 = 6f + 6h - fh - hf$. Multiplying that by f on the right, one has that $6f = 5hf - fhf$. Multiplying this by h on the left, one has that $hf = -hfhf$. Similarly, by the same procedure, $fh = -fhfh$. However, if we multiply $6f = 5hf - fhf$ by f on the left, it will follow that $6f = 4fhf$ and, by symmetry, $6h = 4hfh$. Therefore, by what we have shown so far, $4hf = -4hfhf = -(4hfh)f = -6hf$, that is, $10hf = 10fh = 0$. Furthermore, $12f = 10hf - 2fhf = -2fhf$ which gives that $24f = -4fhf = -6f$ and hence that $30f = 0$. Analogously, $30h = 30e = 0$ and so $120 = 30e + 30f + 30h = 0$ providing us with the desired relation that $2^3.3.5 = 0$.

Writing now $4 = -e-f-h$, we have $-4 = e+f+h$ and $1-e = 5+f+h$. Multiplying this subsequently on the left and on the right, we get that $ef+eh = fe+he = -5e$. Besides, the squaring of $-4 = e+f+h$ leads us to $20 = -10e+fh+hf = -10(-4-f-h)+fh+hf$, i.e., to $-20 = 10f+10h+fh+hf$. The multiplication by f on the right tells us that $-30f = 11hf+fhf$ and the multiplication of the last equality by h on the left gives that $-41hf = hfhf$. In a way of similarity, we have that $-41fh = fhfh$. However, if we multiply $-30f = 11hf + fhf$ by f on the left, we will come to $-30f = 12fhf$ and, by symmetry, $-30h = 12hfh$. Consequently, by what we have proved thus far, $-41.12hf = 12hfhf = (12hfh)f = -30hf$, that is, $462hf = 462fh = 0$. Further, $-42.30f = 42.11hf + 42fhf = 42fhf$ whence $-42.60f = 84fhf = -210f$ and so $30.77f = 0$. Analogically, $30.77h = 30.77e = 0$. By making use of this, we finally infer that $30.77.(-4) = 30.77e + 30.77f + 30.77h = 0$ and so $4.30.77 = 0$ giving up the wanted relation that $2^3.3.5.7.11 = 0$. □

One more comment is that the field $\mathbb{Z}_{11}$ is definitely *not* of this type, however. So, it is an intriguing question of whether or not it could be eventually canceled from the above decomposition.

We close this subsection with the following main decomposing result, which states thus:

Theorem 3. *Let R be a ring. Then the next three points hold:*

(1) *If all elements of R are (either) a sum of three idempotents or a minus sum of two idempotents, then R is decomposable as $R \cong R_1 \times R_2 \times R_3$, where either $R_1 = \{0\}$ or R_1 is a ring of characteristic at most 4, either $R_2 = \{0\}$ or R_2 is a ring of characteristic at most 81, either $R_3 = \{0\}$ or R_3 is a ring of characteristic exactly 5, and all three rings R_1, R_2, R_3 are of the same type as R.*

(2) *If all elements of R are (either) a sum of three idempotents or a difference of two idempotents, then R is decomposable as $R \cong R_1 \times R_2 \times R_3$, where either $R_1 = \{0\}$ or R_1 is a ring of characteristic at most 4, either $R_2 = \{0\}$ or R_2 is a ring of characteristic precisely 3, either $R_3 = \{0\}$ or R_3 is a ring of characteristic precisely 5, and all three rings R_1, R_2, R_3 are of the same type as R.*

(3) *If all elements of R are (either) a sum or a minus sum of three idempotents, then R is decomposable as $R \cong R_1 \times R_2 \times R_3 \times R_4 \times R_5$, where either $R_1 = \{0\}$ or R_1 is a ring of characteristic at most 8, either $R_2 = \{0\}$ or R_2 is a ring of characteristic 3, either $R_3 = \{0\}$ or R_3 is a ring of characteristic 5, either $R_4 = \{0\}$ or R_4 is a ring of characteristic 7, either $R_5 = \{0\}$ or R_5 is a ring of characteristic 11, and all five rings R_1, R_2, R_3, R_4, R_5 are of the same type as R.*

Proof. With the aid of the decompositions given in Lemmas 2, 3, 4, we may successfully apply the classical Chinese Remainder Theorem to get the desired decompositions into more simple direct factors of corresponding characteristic, as wanted.

The second part of the proof about the inheritance of the properties of the decomposing factors R_1, R_2, R_3, R_4, R_5 same as these of R follows by routine element-wise component arguments, so we leave the details and leave their inspection to the interested reader for a direct check. □

In conclusion, it is worth to notice that similar things to these considered above could be considered for the triangular matrix ring $\mathbb{T}_n(R)$ as well.

2.3. Questions and problems.

In regard to Proposition 1, we are able to pose the following:

Conjecture 1. *For all $n \geq 1$, the equality isn$(\mathbb{M}_n(\mathbb{Z}_4)) = 4$ is fulfilled.*

It is worthwhile noticing that the matrix ring $\mathbb{M}_n(\mathbb{Z}_4)$ is also feebly nil-clean in the sense of [3]. In connection with Theorem 2, one may ask the following:

Conjecture 2. *For all $n \geq 1$, any matrix in $\mathbb{M}_n(\mathbb{Z}_4)$ is a sum of a nilpotent matrix of order at most 4 and a potent matrix.*

It is worth noticing that in [28, Example 3.2] was established that every matrix in $\mathbb{M}_n(\mathbb{Z}_4)$ is a sum of a nilpotent matrix of order not exceeding 8 and an idempotent matrix.

We end our exploration with a few more queries of interest:

Problem 1. *If A is a nilpotent matrix over a ring R, is there a power k depending on A such that $tr(A)^k$ is a sum of commutators? In particular, if R is finite, is $tr(A)^k$ a sum of nilpotents?*

For a more account concerning this query the interested reader may see [1] and [20], respectively.

Problem 2. *If R is a finite ring, is then the trace of any nilpotent matrix in $\mathbb{M}_n(R)$ a sum of nilpotents? In particular, does it follow that the inequality $\mathrm{sntr}(\mathbb{M}_n(R)) \leq \omega$ is true?*

In view of Problem 1, this certainly will hold true if every commutator in a finite ring is a sum of nilpotent elements.

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