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**CONSTRUCTION OF A GENERALIZED VORONOI DIAGRAM WITH  
OPTIMAL PLACEMENT OF GENERATOR POINTS BASED ON THE  
THEORY OF OPTIMAL SET PARTITIONING**

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The problem of construction of a generalized Voronoi diagram with optimal placement of a finite number of generator points in a bounded set of  $n$ -dimensional Euclidean space is considered. A method is proposed for solving such a problem based on the formulation of the corresponding continuous problem of optimal partitioning of a set in  $n$ -dimensional Euclidean space with a partition quality criterion that provides the corresponding form of the Voronoi diagram. Further, to solve such a problem, the developed mathematical and algorithmic apparatus is used, the part of which is Shor's  $r$ -algorithm.

The standard (classical) Voronoi diagram [1] of a finite set  $M = \{\tau_1, \tau_2, \dots, \tau_N\} \subset E_n$  of generator points  $\tau_i = (\tau_i^{(1)}, \tau_i^{(2)}, \dots, \tau_i^{(n)})$ ,  $i = 1, 2, \dots, N$  in  $n$ -dimensional Euclidean space  $E_n$  ( $n \geq 2$ ) is the set of Voronoi polytopes

$$Vor(\tau_i) = \{x \in E_n : c(x, \tau_i) \leq c(x, \tau_j), j = 1, 2, \dots, N, j \neq i\}, i = 1, 2, \dots, N \quad (1)$$

of the given points  $\tau_1, \tau_2, \dots, \tau_N$ , where  $c(x, y)$  is a metric in  $E_n$ .

Let  $\Omega$  be a given bounded set of  $E_n$ ,  $\tau_1, \tau_2, \dots, \tau_N$  be a finite set of generator points in  $\Omega$ . In cases when the location of the points  $\tau_1, \tau_2, \dots, \tau_N$  in  $\Omega$  is unknown and they need to be located (selected) in  $\Omega$ , we can introduce another variant of the Voronoi diagram on set  $\Omega \subset E_n$ , which generalizes the standard Voronoi diagram.

By the *Voronoi diagram of a finite number of generator points  $\tau_1, \tau_2, \dots, \tau_N$  optimally located in a bounded set  $\Omega \subset E_n$* , we call the following set of Voronoi polytopes:

$$Vor(\tau_i) = \{x \in \Omega \subset E_n : c(x, \tau_i)/w_i + a_i \leq c(x, \tau_j)/w_j + a_j; i, j = 1, \dots, N, i \neq j\} \quad (2)$$

of points  $\tau_1, \tau_2, \dots, \tau_N$ , where the total weighted distance from the points of set  $\Omega$  to the corresponding generator points  $\tau_1, \tau_2, \dots, \tau_N$  is minimal, so the functional

$$J(\{\tau_1, \dots, \tau_N\}) = \sum_{i=1}^N \int_{Vor(\tau_i)} (c(x, \tau_i)/w_i + a_i) dx \quad (3)$$

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attains the minimum value. Here and in the following, we consider Lebesgue integrals;  $a_i \geq 0$ ,  $w_i > 0$  ( $i = 1, 2, \dots, N$ ) are given numbers (weights).

Note: By specifying the values of the parameters  $a_1, \dots, a_N$ ;  $w_1, \dots, w_N$  and the type of function  $c(x, \tau_i)$  in formula (3), one can obtain various variants of the Voronoi diagram with the optimal placement of generator points (adaptively weighted, multiplicatively weighted ones etc., see [2]).

To construct the Voronoi diagram (2), (3) we describe an approach based on the application of the apparatus of the theory of optimal set partitioning [2, 3].

Let  $\Omega$  be a bounded Lebesgue measurable set in  $n$ -dimensional Euclidean space  $E_n$ . We name a set  $\Omega_1, \dots, \Omega_N$  of Lebesgue measurable subsets of this set by its *possible partitioning*, if  $\bigcup_{i=1}^N \Omega_i = \Omega$ ,  $\text{mes}(\Omega_i \cap \Omega_j) = 0$ ,  $i, j = 1, 2, \dots, N$  ( $i \neq j$ ), where  $\text{mes}(\cdot)$  is a Lebesgue measure. By  $\Sigma_{\Omega}^N$  we denote the class of all possible partitions of a set  $\Omega$  into its non-intersecting subsets that is

$$\Sigma_{\Omega}^N = \left\{ (\Omega_1, \dots, \Omega_N) : \bigcup_{i=1}^N \Omega_i = \Omega, \text{mes}(\Omega_i \cap \Omega_j) = 0, i, j = 1, 2, \dots, N (i \neq j) \right\}.$$

We introduce a functional

$$F(\{\Omega_1, \dots, \Omega_N\}, \{\tau_1, \dots, \tau_N\}) = \sum_{i=1}^N \int_{\Omega_i} (c(x, \tau_i)/w_i + a_i) dx, \quad (4)$$

where  $c(x, \tau_i)$  is a given real function bounded on  $\Omega \times \Omega$  and measurable by argument  $x = (x^{(1)}, \dots, x^{(n)}) \in \Omega$  for any fixed  $\tau_i = (\tau_i^{(1)}, \dots, \tau_i^{(n)}) \in \Omega$  for each  $i = 1, 2, \dots, N$ ;  $a_i \geq 0$ ,  $w_i > 0$  ( $i = 1, 2, \dots, N$ ) are given numbers. We assume that the measure of the set of boundary points of subsets  $\Omega_i$ ,  $i = 1, 2, \dots, N$  is equal to zero.

**Problem A.** To find 
$$\min_{\substack{\{\Omega_1, \dots, \Omega_N\} \in \Sigma_{\Omega}^N, \\ \{\tau_1, \dots, \tau_N\} \in \Omega^N}} F(\{\Omega_1, \dots, \Omega_N\}, \{\tau_1, \dots, \tau_N\}),$$

where functional  $F(\{\Omega_1, \dots, \Omega_N\}, \{\tau_1, \dots, \tau_N\})$  is defined in (4); coordinates  $\tau_i^{(1)}, \dots, \tau_i^{(n)}$  of centers  $\tau_i = (\tau_i^{(1)}, \dots, \tau_i^{(n)}) \in \Omega_i$ ,  $i = 1, 2, \dots, N$  are unknown in advance and must be determined.

Problem A is solvable on  $\Sigma_{\Omega}^N \times \Omega^N$ , as a special case of the continuous single-product problem considered in [3], for the optimal partitioning of a set  $\Omega \subset E_n$  into its subsets  $\Omega_1, \dots, \Omega_N$  (among which there may be empty ones) without constraints, with finding coordinates of these subsets' centers  $\tau_1, \dots, \tau_N$ , respectively.

A pair  $(\{\Omega_1^*, \dots, \Omega_N^*\}, \{\tau_1^*, \dots, \tau_N^*\})$  delivering the least, or minimum, value of the functional (4) on a set  $\Sigma_{\Omega}^N \times \Omega^N$  is called an *optimal solution* to Problem A. Wherein, in Problem A, we name a partition  $\{\Omega_1^*, \dots, \Omega_N^*\} \in \Sigma_{\Omega}^N$  by an *optimal partitioning* of set  $\Omega \subset E_n$  into  $N$  subsets, and a set  $\tau^* = \{\tau_1^*, \dots, \tau_N^*\} \in \Omega^N$  of centers  $\tau_i^* \in \Omega_i^*$ ,  $i = 1, 2, \dots, N$  by *optimal centers* of subsets  $\Omega_i^*$  in Problem A.

Let  $\lambda_i(x) = \begin{cases} 1, & x \in \Omega_i, \\ 0, & x \in \Omega \setminus \Omega_i \end{cases}$  be characteristic functions of subsets  $\Omega_i \subset \Omega$  ( $i = 1, \dots, N$ ).

Following the results of [2, 3], the components of characteristic vector function  $\lambda^*(x) = (\lambda_1^*(x), \dots, \lambda_i^*(x), \dots, \lambda_N^*(x))$  corresponding to an optimal partitioning  $\{\Omega_1^*, \dots, \Omega_i^*, \dots, \Omega_N^*\} \in$

$\sum_{\Omega}^N$  to Problem A for  $i = 1, \dots, N$  and for almost all  $x \in \Omega$  are defined as

$$\lambda_i^*(x) = \begin{cases} 1, & \text{if } c(x, \tau_i^*)/w_i + a_i \leq c(x, \tau_j^*)/w_j + a_j, \\ & \text{for a.a. } x \in \Omega, \quad j = 1, \dots, N \quad (j \neq i), \quad \text{then } x \in \Omega_i^*, \\ 0, & \text{in other cases,} \end{cases} \quad (5)$$

where  $\tau^* = (\tau_1^*, \dots, \tau_i^*, \dots, \tau_N^*) \in \Omega^N$  is an optimal solution to the problem

$$G(\tau) = \int_{\Omega} \min_{i=1, \dots, N} [c(x, \tau_i)/w_i + a_i] dx \rightarrow \min_{\tau \in \Omega^N}. \quad (6)$$

Let us present an algorithm to solve Problem A, which is based on the mathematical apparatus from [2, 3] and on one variant of the generalized gradient descent method with space expansion in the direction of the difference of two successive generalized antigradients (or the so-called Shor's  $r$ -algorithm [4]). In the iterative formula of the  $r$ -algorithm

$$\tau^{[k+1]} = \tau^{[k]} - h_k B_{k+1}^{\tau} [B_{k+1}^{\tau}]^T g_G(\tau^{[k]}), \quad k = 0, 1, \dots, \quad (7)$$

$B_{k+1}^{\tau}$  is an operator that maps the transformed space into the main space  $E_N$  (wherein,  $B_0^{\tau} = I$  is the identity matrix);  $h_k$  is a step factor, which is chosen from the minimum's condition of function  $G(\tau)$  in direction  $B_{k+1}^{\tau} [B_{k+1}^{\tau}]^T g_G(\tau^{[k]})$ ;  $g_G(\tau^{[k]})$  is the generalized gradient of function  $G(\tau)$  at a point  $\tau^{[k]}$ .

We apply the  $r$ -algorithm in the  $H$ -form [4] ( $H_k$  is a symmetric matrix such that  $H_k = B_k B_k^T$ ), for which iterative formula (7) has the form

$$\tau^{[k+1]} = \tau^{[k]} - h_k H_{k+1} g_G(\tau^{[k]}) \left/ \sqrt{(H_{k+1} g_G(\tau^{[k]}), g_G(\tau^{[k]}))}, \quad k = 0, 1, \dots,$$

where

$$H_{k+1} = H_k + (1/\alpha_k^2 - 1) \frac{H_k \Delta_k \Delta_k^T H_k}{(H_k \Delta_k, \Delta_k)}, \quad \Delta_k = g_G(\tau^{[k]}) - g_G(\tau^{[k-1]}).$$

We take the space expansion's coefficient  $\alpha_k$ , which equal to 3, and we apply for finding step factor  $h_k$  the adaptive adjustment method described in [4].

We define the  $i$ -th component of the vector of generalized gradient  $g_G^{\tau}(\tau) = (g_G^{\tau_1}(\tau), \dots, g_G^{\tau_i}(\tau), \dots, g_G^{\tau_N}(\tau))$  of function  $G(\tau)$  from (6) as follows:

$$g_G^{\tau_i}(\tau) = \int_{\Omega} g_c^{\tau_i}(x; \tau) \lambda_i(x) dx, \quad \tau \in \Omega^N, \quad i = 1, \dots, N, \quad (8)$$

where  $g_c^{\tau_i}(x, \tau)$  is the  $i$ -th component of the vector of the generalized gradient  $g_A^{\tau}(x, \tau)$  of function  $c(x, \tau)$  for  $x \in \Omega$ ,  $\tau \in \Omega^N$ .

*Algorithm.*

Step 1. We enclose domain  $\Omega$  in an  $n$ -dimensional parallelepiped  $\Pi$  whose sides are parallel to the axes of the Cartesian coordinate system. We cover parallelepiped  $\Pi$  with a rectangular grid and take the initial approximation  $\tau = \tau^{[0]}$ . Then we calculate values of  $\lambda^{[0]}(x)$  at the grid nodes by formulas (5) for  $\tau = \tau^{[0]}$  and we also calculate values of  $g_G(\tau)$  by formula (8) for  $\lambda(x) = \lambda^{[0]}(x)$ ,  $\tau = \tau^{[0]}$ . We select the initial test step factor  $h_0 > 0$  of the  $r$ -algorithm and find

$$\tau^{[1]} = P_{\Pi} \left( \tau^{[0]} - h_0 H_1 g_G(\tau^{[0]}) \left/ \sqrt{(H_1 g_G(\tau^{[0]}), g_G(\tau^{[0]}))} \right. \right),$$

where  $P_{\Pi}$  is a projection operator on  $\Pi$ . We pass to the second step.

Suppose that after  $k$  steps of the algorithm we have obtained certain values  $\tau^{[k]}$ ,  $\lambda^{[k-1]}(x)$  at the grid nodes ( $k = 1, 2, \dots$ ). Let us describe the  $(k+1)$ -th step of the algorithm.

Step  $k+1$  ( $k = 1, 2, \dots$ ).

1. We calculate  $\lambda^{[k]}(x)$  at the grid nodes according to formulas (5) for  $\tau = \tau^{[k]}$ .
2. We find values of  $g_G(\tau)$  according to formulas (8) for  $\lambda(x) = \lambda^{[k]}(x)$ ,  $\tau = \tau^{[k]}$ .
3. We carry out the  $(k+1)$ -th iteration of the  $r$ -algorithm by the formula

$$\tau^{[k+1]} = P_{\Pi} \left( \tau^{[k]} - h_k H_{k+1} g_G(\tau^{[k]}) / \sqrt{(H_{k+1} g_G(\tau^{[k]}), g_G(\tau^{[k]}))} \right),$$

4. If the condition

$$\|\tau^{[k+1]} - \tau^{[k]}\| \leq \varepsilon, \varepsilon > 0 \quad (9)$$

does not hold, then we go to the  $(k+2)$ -th step of the algorithm, otherwise, we go to the point 5.

5. We assume that  $\lambda^*(x) = \lambda^{[l]}(x)$ ,  $\tau^* = \tau^{[l]}$ , where  $l$  is the iteration number at which condition (9) holds true.

6. We calculate the optimal value of objective function  $G(\tau)$  from (6) at  $\tau = \tau^*$ .

*The algorithm is described.*

Thus, based on the mathematical and algorithmic apparatus of the theory of continuous problems of optimal set partitioning, a generalized Voronoi diagram can be constructed with the optimal placement of generator points in a bounded set of  $n$ -dimensional Euclidean space. For the considered type of a generalized Voronoi diagram, the described construction method is proposed for the first time, and its advantages compared to the algorithms for constructing other generalized Voronoi diagrams known in the scientific literature are confirmed by the results of practical implementations.

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