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A NOTE ON POWER OF MEROMORPHIC FUNCTION AND ITS SHIFT OPERATOR OF CERTAIN HYPER-ORDER SHARING ONE SMALL FUNCTION AND A VALUE

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In this paper, we obtain two results on n-th power of a meromorphic function and its shift operator sharing a small function together with a value which improve and complement some earlier results. In particular, more or less we have improved and extended two results of Qi-Yang [Meromorphic functions that share values with their shifts or their n-th order differences, Analysis Math., 46(4)2020, 843-865] by dispelling the superfluous conclusions in them.

1. Introduction and definitions. Throughout the paper, a meromorphic function will always mean meromorphic in the whole complex plane $\mathbb{C}$. We will use the standard notations in Nevanlinna theory of meromorphic functions such as $m(r,f)$, $N(r,f)$ $(N(r,\infty;f))$, $N(r,\frac{1}{f-a})$ $(N(r,a;f))$, $T(r,f)$. By $S(r,f)$ we mean a quantity satisfying $S(r,f)=o(T(r,f))$ as $r\longrightarrow\infty$ outside of a possible exceptional set E of finite linear measure. (see [4], [10]). With the help of the standard notations we also would like to recall some important terms namely order and hyper-order of f respectively defined as follows

$$\rho(f)=\limsup_{r\longrightarrow\infty}\frac{\log T(r,f)}{\log r},\quad \rho_2(f)=\limsup_{r\longrightarrow\infty}\frac{\log\log T(r,f)}{\log r}.$$

We say that $a(z)$ is *a small function* of $f(z)$ if $T(r,a)=S(r,f)$. We denote by $S(f)$ the set of all small functions compared to $f(z)$. Now we recall the following definition. For a non-constant meromorphic function f and $a\in\mathbb{C}$, let

$$E_f(a)=\{(z,p)\in\mathbb{C}\times\mathbb{N}: f(z)=a \text{ with multiplicity } p\}$$
$$\big(\overline{E}_f(a)=\{(z,1)\in\mathbb{C}\times\mathbb{N}: f(z)=a\}\big).$$

Then we say f, g *share the value* a CM (IM) if $E_f(a)=E_g(a)$ $\big(\overline{E}_f(a)=\overline{E}_g(a)\big)$. For $a=\infty$, we define $E_f(\infty):=E_{1/f}(0)$ $\big(\overline{E}_f(\infty):=\overline{E}_{1/f}(0)\big)$.

Especially, for $a(z)\in S(f)$, if $f-a(z)$ and $g-a(z)$ share 0 IM, then we denote by $\overline{N}_{(k_1,k_2)}(r,0,f-a(z);g-a(z))$, the reduced counting function of common zeros of $f-a(z)$ and $g-a(z)$ with multiplicities k_1 and k_2 respectively. Letting $c\in\mathbb{C}\setminus\{0\}$, we define the shift of $f(z)$ by $f(z+c)$.

2. Auxiliary and main results. In 2010, concerning set sharing for a finite order meromorphic function with its shift operator, Qi et al. [7] investigated the following theorem:

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Theorem A ([7])**.** *Let $f(z)$ be a non-constant meromorphic function of finite order, $n \geq 7$ be an integer and let $F = f^n$. If $F(z)$ and $F(z+c)$ share $a(z) \in S(f) \setminus \{0\}$ and ∞ CM, then $f(z) = tf(z+c)$, for a constant t that satisfies $t^n = 1$.*

Two years later, Qi et al. [8] were able to reduce the cardinality of n in Theorem A from 7 to 4. In [8] the following result was proved.

Theorem B ([8])**.** *Let $f(z)$ be a non-constant meromorphic function of finite order, $a(z) \in S(f) \setminus \{0\}$ be a periodic function with period c, $n \geq 4$ be an integer, and let $F = f^n$. If $F(z)$ and $F(z+c)$ share $a(z)$ and ∞ CM, then $f(z) = tf(z+c)$, for a constant t that satisfies $t^n = 1$.*

In 2017, using another method, Lu-Han [6] further reduced the cardinality of n up to 3.

Theorem C ([6])**.** *Let $f(z)$ be a non-constant meromorphic function of finite order. If $f^3(z)$ and $f^3(z+c)$ share 1, ∞ CM, then $f(z) = t_1 f(z+c)$, where t_1 satisfy $t_1^3 = 1$.*

Recently, adopting the same procedure as in Theorem C, for meromorphic function of $\rho_2 < 1$, Qi-Yang [9] obtained the following theorem:

Theorem D ([9])**.** *Let $f(z)$ be a non-constant meromorphic function of $\rho_2(f) < 1$, $n \geq 3$ be an integer and $a\ (\neq 0) \in \mathbb{C}$. If $f^n(z)$ and $f^n(z+c)$ share a and ∞ CM, then $f(z) = t_1 f(z+c)$ or $f(z) = t_2 f(z+2c)$, where t_1 and t_2 satisfy $t_i^n = 1$ $(i = 1, 2)$.*

Considering $\rho_2 < 1$, using similar method as used in Theorem B, it is easy to prove the following result, which actually shows that for $n \geq 4$, only first conclusion of Theorem D occurs.

Theorem 1. *Let $f(z)$ be a non-constant meromorphic function of $\rho_2(f) < 1$, $a(z) \in S(f) \setminus \{0\}$ be a periodic function with period $c\ (\neq 0)$, $n \geq 4$ be an integer. If $f^n - a(z)$ and $f^n(z+c) - a(z)$ share 0 CM and $f(z)$, $f(z+c)$ share ∞ CM, then $f(z) \equiv t_1 f(z+c)$, where t_1 satisfies $t_1^n = 1$.*

The following corollary immediately holds.

Corollary 1. *Let $f(z)$ be a non-constant entire function of $\rho_2(f) < 1$, $a(z) \in S(f) \setminus \{0\}$ be a periodic function with period $c\ (\neq 0)$, $n \geq 3$ be an integer. If f^n and $f^n(z+c)$ share $a(z)$ CM, then $f(z) \equiv t_1 f(z+c)$, where t_1 satisfies $t_1^n = 1$.*

As our prime motto is to get the uniqueness result and discard the more conclusions, thereby we investigated about the nature of conclusions in Theorem D.

Remark 1. Suppose, conclusion 1 : $f(z) = t_1 f(z+c)$ where t_1 satisfy $t_1^n = 1$,
conclusion 2 : $f(z) = t_2 f(z+2c)$ where t_2 satisfy $t_2^n = 1$.

We can easily show that conclusion 1 implies conclusion 2 for all $n \geq 1$. Suppose conclusion 1 holds. Then $f(z+c) = t_1 f(z+2c) \implies f(z) = t_1^2 f(z+2c)$ where t_1 satisfies $t_1^n = 1$. Obviously $f(z) = t_2 f(z+2c)$ where $t_1^2 = t_2$ which satisfies $(t_1^2)^n = 1$. Hence conclusion 2 holds.

For $n = 1$, conclusion 2 implies conclusion 1 only if $f(z)$ is a function of period c. Next we consider the case $n = 2$. Let $f(z) = \sqrt{2}\sin(\frac{\pi z}{2c})$. Then $f(z+c) = \sqrt{2}\cos(\frac{\pi z}{2c})$ and $f(z+2c) = -\sqrt{2}\sin(\frac{\pi z}{2c})$. Though f and $f(z+c)$ share the set $\{1, -1\}$ CM and conclusion 2 holds but conclusion 1 does not hold.

In view of Remark 1 we know that the second conclusion of Theorem D is enough to concede but it remains an open question about the validity of Theorem 1 for the case $n = 3$ under the same conclusion. But unfortunately we could not succeed.

Regarding sharing a set of two small functions, by a finite order entire function with its shift operator, we recall a result of [5].

Theorem E ([5])**.** *Let $f(z)$ be a transcendental entire function of finite order, $c \in \mathbb{C}\backslash\{0\}$, and let $a(z) \in \mathbb{S}(f)$ be a non-vanishing periodic entire function with period c. If $f(z)$ and $f(z+c)$ share the set $\{a(z), -a(z)\}$ CM, then $f(z)$ must be take one of the following conclusions: 1. $f(z+c) = \pm f(z)$; 2. $f(z) = \frac{h_1(z)+h_2(z)}{2}$, where $\frac{h_1(z+c)}{h_1(z)} = -e^{\gamma}$, $\frac{h_2(z+c)}{h_2(z)} = e^{\gamma}$, $h_1(z)h_2(z) = (a(z))^2\left(1 - e^{-2\gamma}\right)$ and γ is a polynomial.*

For an entire function with $\rho_2(f) < 1$ sharing set with its shift operator Qi-Yang [9] investigated the following theorem:

Theorem F ([9])**.** *Let $f(z)$ be a non-constant entire function of $\rho_2(f) < 1$, $n \geq 2$ be an integer and $a\ (\neq 0) \in \mathbb{C}$. If $f^n(z)$ and $f^n(z+c)$ share a CM, then $f(z) = t_1 f(z+c)$ or $f(z) = t_2 f(z+2c)$, where t_1 and t_2 satisfy $t_i^n = 1$ $(i = 1, 2)$.*

Remark 2. Using two basic lemmas [see Lemmas 1, 3] related to $\rho_2(f) < 1$ one can easily prove that for $\rho_2(f) < 1$, Theorem E is also valid, the only difference is γ, will be an entire function with $\rho(\gamma) < 1$. For the case $n = 2$ in Theorem F, by a simple calculation, we can show that the conclusions of Theorem E are same as in Theorem F. For sake of convenience we explain it. Since f is entire function of $\rho_2(f) < 1$ and f, $f(z+c)$ share the set $\{a(z), -a(z)\}$ CM, so

$$\frac{(f(z+c) - a(z))(f(z+c) + a(z))}{(f - a(z))(f + a(z))} = e^{2\gamma(z)},$$

γ being an entire function with $\rho(\gamma) < 1$. From conclusion (2) of Theorem E we have $h_1(z+c)h_2(z+c) = -e^{2\gamma(z)}h_1(z)h_2(z)$. i.e., $a^2(z+c)(1-e^{-2\gamma(z+c)}) = -e^{2\gamma(z)}a^2(z)(1-e^{-2\gamma(z)})$. Since $a(z)$ is non-vanishing periodic function with period c, so the above implies $e^{2\gamma(z)+2\gamma(z+c)} = 1$, that yields $2\gamma(z+c) + 2\gamma(z) = 2k\pi i$, where k is an integer. If γ is transcendental, so by Lemma 5, stated afterwards, $\rho(\gamma) \geq 1$, which is not possible. Therefore γ must be constant and so $e^{4\gamma} = 1$. i.e., $e^{2\gamma} = \pm 1$. So when $e^{2\gamma} = 1$ we get the first conclusion of Theorem F, where as when $e^{2\gamma} = -1$ we can have $f^2(z+c) = 2a^2(z) - f^2(z)$, i.e., $f^2(z+2c) = f^2(z)$, which gives the second conclusion of the same theorem.

When $n \geq 3$, in view of Remark 1 from Theorems F and Corollary 1, for entire function of hyper-order < 1, it is clear that conclusion 1 and conclusion 2 are equivalent. As in Corollary 1 we obtain conclusion 1 under the more generalized sharing structure, so it is an improvement of Theorem F for $n \geq 3$. Also from the discussion in Remark 1, we know that in Theorem F, for $n = 2$, conclusion 1 is no longer required where as from Remark 2 we know this case can be obtained from Theorem E under some manipulations of the previous results.

We also observe that the first conclusion in Theorem F is more specific as it indicates the straight forward relation between one function and its shift operator, natural questions appear:

i) Is it possible to omit the second conclusion?

ii) What happens when the CM sharing is changed to IM sharing?

In connection to these two questions we will show that at the expense of allowing the sharing of the set 0 IM the relaxation of sharing from CM to IM in Theorems E, F is achievable. With regard to this we have the following theorem:

Theorem 2. *Let $n \geq 2$, f be an entire function. If f^n and $f^n(z+c)$ share $a(z)$, 0 IM, then $f(z) \equiv t_1 f(z+c)$, where t_1 satisfies $t_1^n = 1$.*

By a well-known example, for $n = 2$ we can show that 0 sharing in Theorem 2 can not be removed.

Example 1. Let $f(z) = \sin(\frac{\pi z}{2c})$. Then $f(z+c) = \cos(\frac{\pi z}{2c})$. We know that f and $f(z+c)$ share the set $\{\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\}$ CM but not share the value 0 and $f(z+c) \not\equiv \pm f(z)$.

3. Lemmas. Now we need the following lemmas to proceed further.

Lemma 1 ([3])**.** *Let $f(z)$ be a meromorphic function of $\rho_2(f) < 1$ and $c \in \mathbb{C} \setminus \{0\}$. Then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = S(r, f).$$

Lemma 2 ([3])**.** *Let $T : [0, +\infty) \to [0, +\infty)$ be a non-decreasing continuous function, and let $s \in (0, +\infty)$. If the hyper-order of T is strictly less than 1, i.e.,*

$$\limsup_{r \longrightarrow \infty} \frac{\log \log T(r)}{\log r} = \rho_2 < 1,$$

and $\delta \in (0, 1 - \rho_2)$, then $T(r+s) = T(r) + o(\frac{T(r)}{r^\delta})$, as $r \to \infty$ outside of a set of finite logarithmic measures.

Using this lemma by a simple alteration of the result for finite order meromorphic functions in [2], one can have the following lemma.

Lemma 3. *Let $f(z)$ be a meromorphic function of $\rho_2(f) < 1$, then we have*

$$N(r, f(z+c)) = N(r, f) + S(r, f), \quad T(r, f(z+c)) = T(r, f) + S(r, f).$$

Lemma 4 ([11])**.** *Let $f(z)$ be a non constant meromorphic function in the complex plane, and let $R(f) = \frac{P(f)}{Q(f)}$, where $P(f) = \sum_{k=0}^{p} a_k(z) f^k$, $Q(f) = \sum_{j=0}^{q} b_j(z) f^j$ are two mutually prime polynomials in f. If the coefficients $a_k(z)$ for $k \in \{0, 1, \ldots, p\}$ and $b_j(z)$ for $j \in \{0, 1, \ldots, q\}$ are small functions of f with $a_p(z) \not\equiv 0$ and $b_q(z) \not\equiv 0$, then*

$$T(r, R(f)) = \max\{p, q\} T(r, f) + S(r, f).$$

Lemma 5 ([1])**.** *Let $h_2(z)$ $(\not\equiv 0)$, $h_1(z)$, $F(z)$ be polynomials, c_2, c_1 $(\neq c_2)$ be constants. Suppose that $f(z)$ be a transcendental meromorphic solution of the difference equation*

$$h_2(z) f(z+c_2) + h_1(z) f(z+c_1) = F(z).$$

Then, $\rho(f) \geq 1$.

4. Proofs of the theorems.

Proof of Theorem 1. Let $F = f^n$. As $\rho_2(f) < 1$, so obviously $\rho_2(F) < 1$ and by Lemma 3, $\rho_2(F(z+c)) < 1$. Again since F and $F(z+c)$ share $a(z)$ CM and ∞ CM, therefore

$$\frac{F(z+c) - a(z)}{F - a(z)} = e^{\delta(z)}, \tag{1}$$

where $\delta(z)$ is an entire function. Now, by Lemma 1 we obtain that

$$T(r,e^{\delta(z)})=m(r,e^{\delta(z)})=m\left(r,\frac{F(z+c)-a(z+c)}{F-a(z)}\right)=S(r,F).$$

Rewriting (1) we have $F(z+c)=e^{\delta(z)}[F-a(z)(1-e^{-\delta(z)})]$. If possible let $e^{\delta(z)}\not\equiv 1$. Then by the Second Fundamental Theorem for small functions and using Lemma 3 we deduce that

$$nT(r,f)=T(r,F)\le \overline{N}(r,F)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{F-a(z)(1-e^{-\delta(z)})}\right)+S(r,F)\le$$
$$\le \overline{N}(r,F)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{F(z+c)}\right)+S(r,F)\le \overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f}\right)+$$
$$+\overline{N}\left(r,\frac{1}{f(z+c)}\right)+S(r,f)\le 2T(r,f)+T(r,f(z+c))+S(r,f)\le 3T(r,f)+S(r,f),$$

which contradicts to $n\ge 4$. Therefore $e^{\delta(z)}\equiv 1$, which yields $F(z+c)\equiv F$. i.e., $f(z)\equiv t_1f(z+c)$, where t_1 satisfies $t_1^n=1$. □

Proof of Corollary 1. Proceeding in a similar way as used to prove Theorem 1, we have

$$nT(r,f)\le \overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f(z+c)}\right)+S(r,f).$$

Since f is entire, so here $nT(r,f)\le 2T(r,f)+S(r,f)$, that contradicts to $n\ge 3$. Hence the conclusion holds. □

Proof of Theorem 2. For $n\ge 2$, suppose $F=f^n$ and $F(z+c)\not\equiv F$. Take

$$\alpha(z)=\frac{P_1(F)[(F(z+c))-F]}{F(F-a(z))},\quad \beta(z)=\frac{P_1(F(z+c))[(F(z+c))-F]}{F(z+c)(F(z+c)-a(z))},$$

where $P_1(F)=a(z)F'-a'(z)F$ and $P_1(F(z+c))$ is defined similarly. Clearly $\alpha(z)\not\equiv 0$ as well as $\beta(z)\not\equiv 0$ as $P(F)$ and $P(F(z+c))$ are not equivalent to 0. On the contrary, if they are equivalent to zero then by a simple integration we can show that $T(r,f)=S(r,f)$ and $T(r,f(z+c))=S(r,f)$, which is not possible.

Now by Lemma 1, we obtain

$$m(r,\alpha(z))=m\left(r,\frac{P_1(F)[(F(z+c))-F]}{F(F-a(z))}\right)\le m\left(r,\frac{P_1(F)}{F-a(z)}\right)+$$
$$+m\left(r,\frac{F(z+c)}{F}-1\right)+O(1)\le m\left(r,\frac{F'a(z)-Fa'(z)}{F-a(z)}\right)+S(r,F)=$$
$$=m\left(r,\frac{a(z)(F'-a'(z))}{F-a(z)}-a'(z)\right)+S(r,F)=S(r,F). \tag{2}$$

Similarly we can obtain

$$m(r,\beta(z))=S(r,F). \tag{3}$$

Let z_0 be a zero of F with multiplicity k_1 such that $a(z_0)\ne 0$. Since F and $F(z+c)$ share the set $\{0\}$ IM, so z_0 is also a zero of $F(z+c)$ with multiplicity k_2 (say). Then as $n\ge 2$, z_0 is zero of $\alpha(z)$ as well as $\beta(z)$ with multiplicity at least $\min\{k_1,k_2\}-1$ (≥ 1). So we can write

$$\overline{N}\Big(r,\frac{1}{F}\Big)\le \overline{N}\Big(r,\frac{1}{\alpha(z)}\Big)+\overline{N}\Big(r,\frac{1}{a(z)}\Big)\le \overline{N}\Big(r,\frac{1}{\alpha(z)}\Big)+S(r,F). \tag{4}$$

Now, let z_1 be a zero of $F-a(z)$ with multiplicity l_1 such that $a(z_1)\neq 0$. Since F and $F(z+c)$ share $a(z)$ IM, so z_1 is also a zero of $F(z+c)-a(z)$ with some multiplicity l_2 (say). Then clearly z_1 is zero of $\alpha(z)$ as well as zero of $\beta(z)$ with multiplicity at least $\min\{l_1,l_2\}-1$ and

$$\alpha(z_1)=l_1\left[\frac{(F(z+c))-F}{z-z_1}\right]_{z=z_1} \text{ and } \beta(z_1)=l_2\left[\frac{(F(z+c))-F}{z-z_1}\right]_{z=z_1}. \tag{5}$$

Thus no zeros of F and $F-a(z)$ are poles of $\alpha(z)$ as well as $\beta(z)$ as long as they are not zeros of $a(z)$. So we have

$$N(r,\alpha(z))\leq \overline{N}\Big(r,\frac{1}{a(z)}\Big)=S(r,F) \text{ and similarly } N(r,\beta(z))=S(r,F). \tag{6}$$

Therefore by (2), (3) and (6) we get

$$T(r,\alpha(z))=S(r,F) \text{ and } T(r,\beta(z))=S(r,F). \tag{7}$$

So from (4) and (7) we have

$$\overline{N}\Big(r,\frac{1}{F}\Big)=S(r,F). \tag{8}$$

By the Second Fundamental Theorem, it follows that, $T(r,F)\leq \overline{N}(r,F)+\overline{N}(r,\frac{1}{F})+\overline{N}(r,\frac{1}{F-a(z)})+S(r,F)\leq \overline{N}(r,\frac{1}{F-a(z)})+S(r,F)\leq T(r,F)+S(r,F)$, i.e.,

$$\overline{N}\Big(r,\frac{1}{F-a(z)}\Big)=T(r,F)+S(r,F). \tag{9}$$

First suppose, for two positive integers l_1 and l_2, $l_2\alpha(z)-l_1\beta(z)\neq 0$. From (5) it can be written that

$$\overline{N}_{(l_1,l_2)}(r,0,F-a(z);F(z+c)-a(z))+S(r,F)\leq$$
$$\leq \overline{N}\Big(r,\frac{1}{l_2\alpha(z)-l_1\beta(z)}\Big)\leq T(r,\alpha(z))+T(r,\beta(z))+S(r,F)=S(r,F)$$

and so from (7) and (9) we have

$$T(r,F)=\overline{N}\Big(r,\frac{1}{F-a(z)}\Big)+S(r,F)=\sum_{l_1+l_2\geq 3}\overline{N}_{(l_1,l_2)}(r,0,F-a(z);F(z+c)-a(z))+$$
$$+S(r,F)\leq \frac{1}{2}\sum_{l_1+l_2\geq 3}\Big[\frac{1}{l_1}N\Big(r,0;F-a(z)\mid\geq l_1\Big)+\frac{1}{l_2}N\left(r,0;F(z+c)-a(z)\mid\geq l_2\right)\Big]+$$
$$+S(r,F)\leq \frac{3}{4}T(r,F)+S(r,F),$$

a contradiction, where by $N\left(r,0;F-a(z)\mid\geq n\right)$ we mean the counting function of zeros of $F-a(z)$ with multiplicity $\geq n$.

Next suppose $l_2\alpha(z)=l_1\beta(z)$. If $l_1=l_2=0$, then $F-a(z)$ and $F(z+c)-a(z)$ has no zeros, and then (9) yields $T(r,F)=S(r,F)$, a contradiction. So let l_1, $l_2\neq 0$. Integrating we have $(\frac{F-a(z)}{F})^{l_2}=A(\frac{F(z+c)-a(z)}{F(z+c)})^{l_1}$, where $A(\neq 0)$ is an integrating constant. By Lemma 3 and Lemma 4, it is obvious that $l_1=l_2$, which follows that there exists a nonzero constant B such that $\frac{F-a(z)}{F}=B\frac{F(z+c)-a(z)}{F(z+c)}$.

Since $F(z+c) \neq F$, so $B \neq 1$. Therefore $\frac{a(z)}{1-B} \neq 0$. Rewriting the above equation we get

$$(F(z+c)-a(z))\Big(F-\frac{a(z)}{1-B}\Big)+\frac{a(z)}{1-B}(F-a(z))=0,$$

i.e., $F-\frac{a(z)}{1-B}=\frac{a(z)}{B-1}\frac{F-a(z)}{F(z+c)-a(z)}$, which yields zeros of $F-\frac{a(z)}{1-B}$ come from either zeros of $a(z)$ or zeros of $F-a(z)$. The second case is possible if $\frac{a(z)}{1-B}$ is Picard's exceptional value. Therefore $\overline{N}\Big(r,\frac{1}{F-\frac{a(z)}{1-B}}\Big)=S(r,F)$. By the Second Fundamental Theorem of small functions we get

$$2T(r,F) \leq \overline{N}(r,F)+\overline{N}\Big(r,\frac{1}{F}\Big)+\overline{N}\Big(r,\frac{1}{F-a(z)}\Big)+\overline{N}\Big(r,\frac{1}{F-\frac{a(z)}{1-B}}\Big)+S(r,f) \leq$$

$$\leq \overline{N}\Big(r,\frac{1}{F}\Big)+\overline{N}\Big(r,\frac{1}{F-a(z)}\Big)+S(r,F),$$

which in view of (8) and (9) is a contradiction. Therefore we must have $F(z+c) \equiv F(z)$. □

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