
In this paper, we obtain two results on n-th power of a meromorphic function and its shift operator sharing a small function together with a value which improve and complement some earlier results. In particular, more or less we have improved and extended two results of Qi-Yang [Meromorphic functions that share values with their shifts or their n-th order differences, Analysis Math., 46(4)2020, 843-865] by dispelling the superfluous conclusions in them.

1. Introduction and definitions. Throughout the paper, a meromorphic function will always mean meromorphic in the whole complex plane \( \mathbb{C} \). We will use the standard notations in Nevanlinna theory of meromorphic functions such as \( m(r,f) \), \( N(r,f) \), \( N(r,\infty;f) \), \( N(r,\frac{1}{f-a}) \), \( N(r,a;f) \), \( T(r,f) \). By \( S(r,f) \) we mean a quantity satisfying \( S(r,f) = o(T(r,f)) \) as \( r \to \infty \) outside of a possible exceptional set \( E \) of finite linear measure. (see [4], [10]). With the help of the standard notations we also would like to recall some important terms namely order and hyper-order of \( f \) respectively defined as follows

\[
\rho(f) = \limsup_{r \to \infty} \frac{\log T(r,f)}{\log r}, \quad \rho_2(f) = \limsup_{r \to \infty} \frac{\log \log T(r,f)}{\log r}.
\]

We say that \( a(z) \) is a small function of \( f(z) \) if \( T(r,a) = S(r,f) \). We denote by \( S(f) \) the set of all small functions compared to \( f(z) \). Now we recall the following definition. For a non-constant meromorphic function \( f \) and \( a \in \mathbb{C} \), let

\[
E_f(a) = \{(z, p) \in \mathbb{C} \times \mathbb{N} : f(z) = a \ \text{with} \ \text{multiplicity} \ \text{p}\}
\]

\[
(\mathcal{E}_f(a) = \{(z, 1) \in \mathbb{C} \times \mathbb{N} : f(z) = a\}).
\]

Then we say \( f, g \) share the value \( a \) CM (IM) if \( E_f(a) = E_g(a) \) (\( \mathcal{E}_f(a) = \mathcal{E}_g(a) \)). For \( a = \infty \), we define \( E_f(\infty) := E_{1/f}(0) \) (\( \mathcal{E}_f(\infty) := \mathcal{E}_{1/f}(0) \)).

Especially, for \( a(z) \) in \( S(f) \), if \( f - a(z) \) and \( g - a(z) \) share 0 IM, then we denote by \( N_{k_1,k_2}(r, 0, f - a(z); g - a(z)) \), the reduced counting function of common zeros of \( f - a(z) \) and \( g - a(z) \) with multiplicities \( k_1 \) and \( k_2 \) respectively. Letting \( c \in \mathbb{C} \setminus \{0\} \), we define the shift of \( f(z) \) by \( f(z + c) \).

2. Auxiliary and main results. In 2010, concerning set sharing for a finite order meromorphic function with its shift operator, Qi et al. [7] investigated the following theorem:

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Theorem A ([7]). Let $f(z)$ be a non-constant meromorphic function of finite order, $n \geq 7$ be an integer and let $F = f^n$. If $F(z)$ and $F(z + c)$ share $a(z)$ in $S(f) \setminus \{0\}$ and $\infty$ CM, then $f(z) = tf(z + c)$, for a constant $t$ that satisfies $t^n = 1$.

Two years later, Qi et al. [8] were able to reduce the cardinality of $n$ in Theorem A from 7 to 4. In [8] the following result was proved.

Theorem B ([8]). Let $f(z)$ be a non-constant meromorphic function of finite order, $a(z) \in S(f) \setminus \{0\}$ be a periodic function with period $c$, $n \geq 4$ be an integer, and let $F = f^n$. If $F(z)$ and $F(z + c)$ share $a(z)$ and $\infty$ CM, then $f(z) = tf(z + c)$, for a constant $t$ that satisfies $t^n = 1$.

In 2017, using another method, Lu-Han [6] further reduced the cardinality of $n$ up to 3.

Theorem C ([6]). Let $f(z)$ be a non-constant meromorphic function of finite order. If $f^3(z)$ and $f^3(z + c)$ share 1, $\infty$ CM, then $f(z) = t_1f(z + c)$, where $t_1$ satisfy $t_1^3 = 1$.

Recently, adopting the same procedure as in Theorem C, for meromorphic function of $\rho_2 < 1$, Qi-Yang [9] obtained the following theorem:

Theorem D ([9]). Let $f(z)$ be a non-constant meromorphic function of $\rho_2(f) < 1$, $n \geq 3$ be an integer and $a$ $(\neq 0) \in \mathbb{C}$. If $f^n(z)$ and $f^n(z + c) - a(z)$ share 0 CM and $f(z)$, $f(z + c)$ share $\infty$ CM, then $f(z) \equiv t_1f(z + c)$, where $t_1$ satisfies $t_1^n = 1$.

Considering $\rho_2 < 1$, using similar method as used in Theorem B, it is easy to prove the following result, which actually shows that for $n \geq 4$, only first conclusion of Theorem D occurs.

Theorem 1. Let $f(z)$ be a non-constant meromorphic function of $\rho_2(f) < 1$, $a(z) \in S(f) \setminus \{0\}$ be a periodic function with period $c$ $(\neq 0)$, $n \geq 4$ be an integer. If $f^n - a(z)$ and $f^n(z + c) - a(z)$ share 0 CM and $f(z)$, $f(z + c)$ share $\infty$ CM, then $f(z) \equiv t_1f(z + c)$, where $t_1$ satisfies $t_1^n = 1$.

The following corollary immediately holds.

Corollary 1. Let $f(z)$ be a non-constant entire function of $\rho_2(f) < 1$, $a(z) \in S(f) \setminus \{0\}$ be a periodic function with period $c$ $(\neq 0)$, $n \geq 3$ be an integer. If $f^n$ and $f^n(z + c)$ share $a(z)$ CM, then $f(z) \equiv t_1f(z + c)$, where $t_1$ satisfies $t_1^n = 1$.

As our prime motto is to get the uniqueness result and discard the more conclusions, thereby we investigated about the nature of conclusions in Theorem D.

Remark 1. Suppose, conclusion 1 : $f(z) = t_1f(z + c)$ where $t_1$ satisfy $t_1^n = 1$, conclusion 2 : $f(z) = t_2f(z + 2c)$ where $t_2$ satisfy $t_2^n = 1$.

We can easily show that conclusion 1 implies conclusion 2 for all $n \geq 1$. Suppose conclusion 1 holds. Then $f(z + c) = t_1f(z + 2c) \implies f(z) = t_1^2f(z + 2c)$ where $t_1$ satisfies $t_1^n = 1$. Obviously $f(z) = t_2f(z + 2c)$ where $t_1^2 = t_2$ which satisfies $(t_2^2)^n = 1$. Hence conclusion 2 holds.

For $n = 1$, conclusion 2 implies conclusion 1 only if $f(z)$ is a function of period $c$. Next we consider the case $n = 2$. Let $f(z) = \sqrt{2}\sin(\frac{\pi z}{2c})$. Then $f(z + c) = \sqrt{2}\cos(\frac{\pi z}{2c})$ and $f(z + 2c) = -\sqrt{2}\sin(\frac{\pi z}{2c})$. Though $f$ and $f(z + c)$ share the set $\{1, -1\}$ CM and conclusion 2 holds but conclusion 1 does not hold.
In view of Remark 1 we know that the second conclusion of Theorem D is enough to
cede but it remains an open question about the validity of Theorem 1 for the case \( n = 3 \)
under the same conclusion. But unfortunately we could not succeed.

Regarding sharing a set of two small functions, by a finite order entire function with its
shift operator, we recall a result of [5].

**Theorem E** ([5]). Let \( f(z) \) be a transcendental entire function of finite order, \( c \in \mathbb{C} \setminus \{0\} \), and
let \( a(z) \in \mathbb{S}(f) \) be a non-vanishing periodic entire function with period \( c \). If \( f(z) \) and \( f(z+c) \)
share the set \( \{a(z), -a(z)\} \) CM, then \( f(z) \) must be take one of the following conclusions:
1. \( f(z+c) = \pm f(z) \); 2. \( f(z) = \frac{h_1(z)+h_2(z)}{2} \), where \( \frac{h_1(z+c)}{h_1(z)} = -e^\gamma \), \( \frac{h_2(z+c)}{h_2(z)} = e^\gamma \), \( h_1(z)h_2(z) = (a(z))^2 (1 - e^{-2\gamma}) \) and \( \gamma \) is a polynomial.

For an entire function with \( \rho_2(f) < 1 \) sharing set with its shift operator Qi-Yang [9]
investigated the following theorem:

**Theorem F** ([9]). Let \( f(z) \) be a non-constant entire function of \( \rho_2(f) < 1 \), \( n \geq 2 \) be an
integer and \( a (\neq 0) \in \mathbb{C} \). If \( f^n(z) \) and \( f^n(z+c) \) share a CM, then \( f(z) = t_1 f(z+c) \) or \( f(z) = t_2 f(z+2c) \), where \( t_1 \) and \( t_2 \) satisfy \( t_i^n = 1 \) (\( i = 1, 2 \)).

**Remark 2.** Using two basic lemmas [see Lemmas 1, 3] related to \( \rho_2(f) < 1 \) one can easily
prove that for \( \rho_2(f) < 1 \), Theorem E is also valid, the only difference is \( \gamma \), will be an entire
function with \( \rho(\gamma) < 1 \). For the case \( n = 2 \) in Theorem F, by a simple calculation, we can
show that the conclusions of Theorem E are same as in Theorem F. For sake of convenience
we explain it. Since \( f \) is entire function of \( \rho_2(f) < 1 \) and \( f, f(z+c) \) share the set \( \{a(z), -a(z)\} \)
CM, so

\[
\frac{(f(z+c) - a(z))(f(z+c) + a(z))}{(f - a(z))(f + a(z))} = e^{2\gamma(z)},
\]

\( \gamma \) being an entire function with \( \rho(\gamma) < 1 \). From conclusion (2) of Theorem E we have \( h_1(z + c)h_2(z + c) = -e^{2\gamma(z)}h_1(z)h_2(z) \), i.e., \( a^2(z+c)(1-e^{-2\gamma(z+c)}) = -e^{2\gamma(z)}a^2(z)(1-e^{-2\gamma(z)}) \). Since
\( a(z) \) is non-vanishing periodic function with period \( c \), so the above implies \( e^{2\gamma(z)+2\gamma(z+c)} = 1 \),
that yields \( 2\gamma(z+c) + 2\gamma(z) = 2k\pi i \), where \( k \) is an integer. If \( \gamma \) is transcendental, so by
Lemma 5, stated afterwards, \( \rho(\gamma) \geq 1 \), which is not possible. Therefore \( \gamma \) must be constant
and so \( e^{4\gamma} = 1 \). i.e., \( e^{2\gamma} = \pm 1 \). So when \( e^{2\gamma} = 1 \) we get the first conclusion of Theorem F,
where as when \( e^{2\gamma} = -1 \) we can have \( f^2(z+c) = 2a^2(z) - f^2(z) \), i.e., \( f^2(z+2c) = f^2(z) \),
which gives the second conclusion of the same theorem.

When \( n \geq 3 \), in view of Remark 1 from Theorems F and Corollary 1, for entire function
of hyper-order \( < 1 \), it is clear that conclusion 1 and conclusion 2 are equivalent. As in
Corollary 1 we obtain conclusion 1 under the more generalized sharing structure, so it is an
improvement of Theorem F for \( n \geq 3 \). Also from the discussion in Remark 1, we know that
in Theorem F, for \( n = 2 \), conclusion 1 is no longer required where as from Remark 2 we
know this case can be obtained from Theorem E under some manipulations of the previous
results.

We also observe that the first conclusion in Theorem F is more specific as it indicates
the straight forward relation between one function and its shift operator, natural questions
appear:

i) Is it possible to omit the second conclusion?

ii) What happens when the CM sharing is changed to IM sharing?
In connection to these two questions we will show that at the expense of allowing the sharing of the set IM the relaxation of sharing from CM to IM in Theorems E, F is achievable. With regard to this we have the following theorem:

**Theorem 2.** Let \( n \geq 2, f \) be an entire function. If \( f^n \) and \( f^n(z+c) \) share \( a(z), 0 \) IM, then \( f(z) \equiv t_1 f(z+c) \), where \( t_1 \) satisfies \( t_1^n = 1 \).

By a well-known example, for \( n = 2 \) we can show that 0 sharing in Theorem 2 can not be removed.

**Example 1.** Let \( f(z) = \sin(\frac{\pi z}{2}) \). Then \( f(z+c) = \cos(\frac{\pi z}{2}) \). We know that \( f \) and \( f(z+c) \) share the set \( \{ \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \} \) CM but not share the value 0 and \( f(z+c) \neq \pm f(z) \).

3. **Lemmas.** Now we need the following lemmas to proceed further.

**Lemma 1 ([3]).** Let \( f(z) \) be a meromorphic function of \( \rho_2(f) < 1 \) and \( c \in \mathbb{C} \setminus \{0\} \). Then
\[
m\left(r, \frac{f(z) + c}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z) + c}\right) = S(r, f).
\]

**Lemma 2 ([3]).** Let \( T : [0, +\infty) \to [0, +\infty) \) be a non-decreasing continuous function, and let \( s \in (0, +\infty) \). If the hyper-order of \( T \) is strictly less than 1, i.e.,
\[
\limsup_{r \to \infty} \frac{\log \log T(r)}{\log r} = \rho_2 < 1,
\]
and \( \delta \in (0, 1 - \rho_2) \), then \( T(r + s) = T(r) + o\left(\frac{T(r)}{r}\right) \), as \( r \to \infty \) outside of a set of finite logarithmic measures.

Using this lemma by a simple alteration of the result for finite order meromorphic functions in [2], one can have the following lemma.

**Lemma 3.** Let \( f(z) \) be a meromorphic function of \( \rho_2(f) < 1 \), then we have
\[
N(r, f(z+c)) = N(r, f) + S(r, f), \quad T(r, f(z+c)) = T(r, f) + S(r, f).
\]

**Lemma 4 ([11]).** Let \( f(z) \) be a non constant meromorphic function in the complex plane, and let \( R(f) = \frac{P(f)}{Q(f)} \), where \( P(f) = \sum_{k=0}^{p} a_k z f^k \), \( Q(f) = \sum_{j=0}^{q} b_j z f^j \) are two mutually prime polynomials in \( f \). If the coefficients \( a_k(z) \) for \( k \in \{0, 1, \ldots, p\} \) and \( b_j(z) \) for \( j \in \{0, 1, \ldots, q\} \) are small functions of \( f \) with \( a_p(z) \neq 0 \) and \( b_q(z) \neq 0 \), then
\[
T(r, R(f)) = \max\{p, q\} T(r, f) + S(r, f).
\]

**Lemma 5 ([11]).** Let \( h_2(z) \neq 0 \), \( h_1(z), F(z) \) be polynomials, \( c_2, c_1 \neq c_2 \) be constants. Suppose that \( f(z) \) be a transcendentinal meromorphic solution of the difference equation
\[
h_2(z) f(z + c_2) + h_1(z) f(z + c_1) = F(z).
\]
Then, \( \rho(f) \geq 1 \).

4. **Proofs of the theorems.**

**Proof of Theorem 1.** Let \( F = f^n \). As \( \rho_2(f) < 1 \), so obviously \( \rho_2(F) < 1 \) and by Lemma 3, \( \rho_2(F(z+c)) < 1 \). Again since \( F \) and \( F(z+c) \) share \( a(z) \) CM and \( \infty \) CM, therefore
\[
\frac{F(z+c) - a(z)}{F - a(z)} = e^{\delta(z)}, \quad (1)
\]
where $\delta(z)$ is an entire function. Now, by Lemma 1 we obtain that
\[
T(r, e^{\delta(z)}) = m(r, e^{\delta(z)}) = m(r, \frac{F(z + c) - a(z + c)}{F - a(z)}) = S(r, F).
\]

Rewriting (1) we have $F(z + c) = e^{\delta(z)}[F - a(z)(1 - e^{-\delta(z)})]$. If possible let $e^{\delta(z)} \neq 1$. Then by the Second Fundamental Theorem for small functions and using Lemma 3 we deduce that
\[
nT(r, f) = T(r, F) \leq N(r, F) + N \left( r, \frac{1}{F} \right) + N \left( r, \frac{1}{F - a(z)(1 - e^{-\delta(z)})} \right) + S(r, F) \leq
\]
\[
\leq N(r, F) + N \left( r, \frac{1}{F} \right) + N \left( r, \frac{1}{F(z + c)} \right) + S(r, F) \leq N(r, F) + N \left( r, \frac{1}{F} \right) +
\]
\[
+ N \left( r, \frac{1}{f(z + c)} \right) + S(r, f) \leq 2T(r, f) + T(r, f(z + c)) + S(r, f) \leq 3T(r, f) + S(r, f),
\]
which contradicts to $n \geq 4$. Therefore $e^{\delta(z)} \equiv 1$, which yields $F(z + c) \equiv F$, i.e., $f(z) \equiv t_1 f(z + c)$, where $t_1$ satisfies $t_1^k = 1$. \hfill \Box

**Proof of Corollary 1.** Proceeding in a similar way as used to prove Theorem 1, we have
\[
nT(r, f) \leq N(r, f) + N \left( r, \frac{1}{f} \right) + N \left( r, \frac{1}{f(z + c)} \right) + S(r, f).
\]
Since $f$ is entire, so here $nT(r, f) \leq 2T(r, f) + S(r, f)$, that contradicts to $n \geq 3$. Hence the conclusion holds. \hfill \Box

**Proof of Theorem 2.** For $n \geq 2$, suppose $F = f^n$ and $F(z + c) \neq F$. Take
\[
\alpha(z) = \frac{P_1(F)[(F(z + c) - F]}{F(F - a(z))}, \quad \beta(z) = \frac{P_1(F(z + c))[F(z + c) - F]}{F(z + c)(F(z + c) - a(z))},
\]
where $P_1(F) = a(z)F' - a'(z)F$ and $P_1(F(z + c))$ is defined similarly. Clearly $\alpha(z) \neq 0$ as well as $\beta(z) \neq 0$ as $P(F)$ and $P(F(z + c))$ are not equivalent to 0. On the contrary, if they are equivalent to zero then by a simple integration we can show that $T(r, f) = S(r, f)$ and $T(r, f(z + c)) = S(r, f)$, which is not possible.

Now by Lemma 1, we obtain
\[
m(r, \alpha(z)) = m \left( r, \frac{P_1(F)[(F(z + c) - F]}{F(F - a(z))} \right) \leq m \left( r, \frac{P_1(F)}{F - a(z)} \right) +
\]
\[
+ m \left( r, \frac{F(z + c)}{F} - 1 \right) + O(1) \leq m \left( r, \frac{F' a(z) - Fa'(z)}{F - a(z)} \right) + S(r, F) =
\]
\[
= m \left( r, \frac{a(z)(F' - a'(z))}{F - a(z)} - a'(z) \right) + S(r, F) = S(r, F). \tag{2}
\]
Similarly we can obtain
\[
m(r, \beta(z)) = S(r, F). \tag{3}
\]
Let $z_0$ be a zero of $F$ with multiplicity $k_1$ such that $a(z_0) \neq 0$. Since $F$ and $F(z + c)$ share the set $\{0\}$ IM, so $z_0$ is also a zero of $F(z + c)$ with multiplicity $k_2$ (say). Then as $n \geq 2$, $z_0$ is zero of $\alpha(z)$ as well as $\beta(z)$ with multiplicity at least $\min\{k_1, k_2\} - 1 (\geq 1)$. So we can write
\[
N \left( r, \frac{1}{F} \right) \leq N \left( r, \frac{1}{\alpha(z)} \right) + N \left( r, \frac{1}{a(z)} \right) \leq N \left( r, \frac{1}{\alpha(z)} \right) + S(r, F). \tag{4}
\]
Now, let \( z_1 \) be a zero of \( F - a(z) \) with multiplicity \( l_1 \) such that \( a(z_1) \neq 0 \). Since \( F \) and \( F(z + c) \) share \( a(z) \) IM, so \( z_1 \) is also a zero of \( F(z + c) - a(z) \) with some multiplicity \( l_2 \) (say). Then clearly \( z_1 \) is zero of \( \alpha(z) \) as well as zero of \( \beta(z) \) with multiplicity at least \( \min\{l_1, l_2\} - 1 \)
and
\[
\alpha(z_1) = l_1 \left[ \frac{(F(z + c) - F)}{z - z_1} \right]_{z = z_1} \quad \text{and} \quad \beta(z_1) = l_2 \left[ \frac{(F(z + c) - F)}{z - z_1} \right]_{z = z_1}. \tag{5}
\]
Thus no zeros of \( F \) and \( F - a(z) \) are poles of \( \alpha(z) \) as well as \( \beta(z) \) as long as they are not zeros of \( a(z) \). So we have
\[
N(r, \alpha(z)) \leq \overline{N}(r, \frac{1}{a(z)}) = S(r, F) \quad \text{and similarly} \quad N(r, \beta(z)) = S(r, F). \tag{6}
\]
Therefore by (2), (3) and (6) we get
\[
T(r, \alpha(z)) = S(r, F) \quad \text{and} \quad T(r, \beta(z)) = S(r, F). \tag{7}
\]
So from (4) and (7) we have
\[
\overline{N}(r, \frac{1}{F}) = S(r, F). \tag{8}
\]
By the Second Fundamental Theorem, it follows that,
\[
T(r, F) \leq \overline{N}(r, F) + \overline{N}(r, \frac{1}{F}) + \overline{N}(r, \frac{1}{F - a(z)}) + S(r, F) \leq \overline{N}(r, \frac{1}{F - a(z)}) + S(r, F) \leq T(r, F) + S(r, F), \] i.e.,
\[
\overline{N}(r, \frac{1}{F - a(z)}) = T(r, F) + S(r, F). \tag{9}
\]
First suppose, for two positive integers \( l_1 \) and \( l_2 \), \( l_2 \alpha(z) - l_1 \beta(z) \neq 0 \). From (5) it can be written that
\[
\overline{N}_{(l_1, l_2)}(r, 0, F - a(z); F(z + c) - a(z) + S(r, F) \leq \overline{N}(r, \frac{1}{l_2 \alpha(z) - l_1 \beta(z)}) \leq T(r, \alpha(z)) + T(r, \beta(z)) + S(r, F) = S(r, F)
\]
and so from (7) and (9) we have
\[
T(r, F) = \overline{N}(r, \frac{1}{F - a(z)}) + S(r, F) = \sum_{l_1 + l_2 \geq 3} \overline{N}_{(l_1, l_2)}(r, 0, F - a(z); F(z + c) - a(z)) +
\]
\[
+ S(r, F) \leq \frac{1}{2} \sum_{l_1 + l_2 \geq 3} \left[ \frac{1}{l_1} N(r, 0; F - a(z) \geq l_1) + \frac{1}{l_2} N(r, 0; F(z + c) - a(z) \geq l_2) \right] +
\]
\[
+ S(r, F) \leq \frac{3}{4} T(r, F) + S(r, F),
\]
a contradiction, where by \( N(r, 0; F - a(z) \geq n) \) we mean the counting function of zeros of \( F - a(z) \) with multiplicity \( \geq n \).

Next suppose \( l_2 \alpha(z) = l_1 \beta(z) \). If \( l_1 = l_2 = 0 \), then \( F - a(z) \) and \( F(z + c) - a(z) \) has no zeros, and then (9) yields \( T(r, F) = S(r, F) \), a contradiction. So let \( l_1, l_2 \neq 0 \). Integrating we have \( (\frac{F - a(z)}{F}) l_2 = A(\frac{F(z + c) - a(z)}{F(z + c)}) l_1 \), where \( A(\neq 0) \) is an integrating constant. By Lemma 3 and Lemma 4, it is obvious that \( l_1 = l_2 \), which follows that there exists a nonzero constant \( B \) such that \( \frac{F - a(z)}{F} = B \frac{F(z + c) - a(z)}{F(z + c)} \).
Since $F(z+c) \neq F$, so $B \neq 1$. Therefore $\frac{a(z)}{1-B} \neq 0$. Rewriting the above equation we get
\[
(F(z+c) - a(z))\left(F - \frac{a(z)}{1-B}\right) + \frac{a(z)}{1-B}(F - a(z)) = 0,
\]
i.e., $F - \frac{a(z)}{1-B} = \frac{a(z)}{B-1} F(z+c) - a(z)$, which yields zeros of $F - \frac{a(z)}{1-B}$ come from either zeros of $a(z)$ or zeros of $F - a(z)$. The second case is possible if $\frac{a(z)}{1-B}$ is Picard’s exceptional value. Therefore $\overline{N}\left(r, \frac{1}{F - \frac{a(z)}{1-B}}\right) = S(r, F)$. By the Second Fundamental Theorem of small functions we get
\[
2T(r, F) \leq \overline{N}(r, F) + \overline{N}\left(r, \frac{1}{F} - \frac{1}{F - a(z)}\right) + \overline{N}\left(r, \frac{1}{F - \frac{a(z)}{1-B}}\right) + S(r, f) \leq
\]
\[
\leq \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{F - a(z)}\right) + S(r, F),
\]
which in view of (8) and (9) is a contradiction. Therefore we must have $F(z+c) \equiv F(z)$. 

REFERENCES

9. Qi X.G., Yang L.Z., Meromorphic functions that share values with their shifts or their n-th order differences, Analysis Math., 46 (2020), 843–865.