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ASYMPTOTICS OF δ-SUBHARMONIC FUNCTIONS OF FINITE ORDER

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For a δ-subharmonic in $\mathbb{R}^m$, $m \geq 2$, function $u = u_1 - u_2$ of finite positive order we found the asymptotical representation of the form

$$u(x) = -I(x, u_1) + I(x, u_2) + O\left(V(|x|)\right), \ x \to \infty,$$

where

$$I(x, u_i) = \int\limits_{|a-x|\leq|x|} K(x,a)d\mu_i(a), \quad K(x,a) = \ln\frac{|x|}{|x-a|}$$

for $m = 2$, $K(x,a) = |x-a|^{2-m} - |x|^{2-m}$ for $m \geq 3$, μ_i is the Riesz measure of the subharmonic function u_i, $V(r) = r^{\rho(r)}$, $\rho(r)$ is a proximate order of u. The obtained result generalizes one theorem of I. F. Krasichkov for entire functions.

In this paper one theorem of I.F. Krasichkov ([6], theorem 1) is generalized to the case of δ-subharmonic in $\mathbb{R}^m$, $m \geq 2$, functions of finite positive order.

Let u be a δ-subharmonic function in $\mathbb{R}^m$, $m \geq 2$, that is $u = u_1 - u_2$, where u_1, u_2 are subharmonic functions. Without loss of generality, we assume that u_1, u_2 are harmonic in a neighborhood of zero and their orders do not exceed the order of u, the Riesz masses of u_1 and u_2 are concentrated on disjoint sets and $u_1(0) = u_2(0) = 0$. We set

$$M(a,r) = \ln(r/|a|),\ L(x,a) = \ln\left(|x|/|x-a|\right)$$

for $m = 2$ and

$$M(a,r) = |a|^{2-m} - r^{2-m},\ L(x,a) = |x-a|^{2-m} - |x|^{2-m}$$

for $m \geq 3$, $x, a \in \mathbb{R}^m$, $r \geq 0$. Let μ_i be a Riesz measure of subharmonic function u_i, $|x| = r$,

$$n(t, u_i) = \mu_i\left(\{h : |h| \leq t\}\right),\ n(t, x, u_i) = \mu_i\left(\{h : |h-x| \leq t\}\right),$$

$d_m = m - 2$, for $m \geq 3$, $d_2 = 1$. We put

$$N(r, u_i) = \int\limits_{|a|\leq r} M(a,r)d\mu_i(a) = d_m\int\limits_0^r n(t,u_i)t^{1-m}dt,$$

$$I(x, u_i) = \int\limits_{|x-a|\leq r} L(x,a)d\mu_i(a) = d_m\int\limits_0^r n(t,x,u_i)t^{1-m}dt.$$

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The quantity $I(x,u_i)$ is called a concentration index of a subharmonic function u_i at the point x [3, p. 227]. The similar term was used for a related concept in case of entire functions by I.F. Krasichkov [6]. In fact, the quantity $I(x,u_i)$ was introduced in the book of B.Ya. Levin [7, p. 164] (see also [5]) before [6].

Let $\rho(r)$ be a proximate order of the function u, that is:

1) $\rho(r) \ge 0$ on $[0,+\infty)$;
2) $\rho(r) \to \rho > 0$ as $r \to +\infty$;
3) the function $\rho(r)$ is continuously differentiable on $[0,+\infty)$;
4) $r\rho'(r)\ln r \to 0$ as $r \to +\infty$;
5) $0 < \varlimsup\limits_{r\to+\infty} T(r,u)/V(r) < +\infty$, with $V(r) = r^{\rho(r)}$,

   where $T(r,u)$ is the Nevanlinna characteristic of the function u, i.e.

$$T(r,u) = \frac{1}{|\mathcal{S}^{m-1}|}\int\limits_{\mathcal{S}^{m-1}} \max\{u_1(rx),u_2(rx)\}d\mathcal{S}(x), \quad 0 \le r < +\infty,$$

   $d\mathcal{S}(x)$ is the sphere area element of $\mathcal{S}^{m-1} = \{x \in \mathbb{R}^m\colon |x| = 1\}$, and $|\mathcal{S}^{m-1}| = 2\pi^{m/2}/\Gamma(m/2)$. If u is a subharmonic function on $\mathbb{R}^m$ $(m \ge 2)$, $u(0) = 0$, then

$$T(r,u) = \frac{1}{|\mathcal{S}^{m-1}|}\int\limits_{\mathcal{S}^{m-1}} u^+(rx)d\mathcal{S}(x), \quad u^+ = \max\{u,0\}, \quad 0 \le r < +\infty.$$

We set $K(x) = \log|x|$ for $m = 2$ and $K(x) = -|x|^{2-m}$ for $m \ge 3$. The function $K(x-a)$ is harmonic in $\mathbb{R}^m$ except the point $x = a$. In particular, if $a \ne 0$ then $K(x-a)$ can be represented as a power series in variables $x_1, x_2, \dots, x_m$, convergent in the neighborhood of the origin. We have

$$K(x-a) = \sum_{\nu=0}^{+\infty} b_\nu(x,a),$$

where for fixed ν and $a \ne 0$ by $b_\nu(x,a)$ we denote a homogeneous harmonic polynomial of $x_1, x_2, \dots, x_m$, whose degree is ν. Let us set

$$K_q(x,a) = K(x-a) - \sum_{\nu=0}^{q} b_\nu(x,a).$$

The Brelot's results ([2, pp.145, 147], Theorems 1, 2) imply the analogue of Lindelöf theorem for subharmonic functions of finite order ([1, p. 9]).

Theorem A (**Analogue of the Lindelöf theorem**)**.** *Let v be a subharmonic function of integer order ρ, $S_1 = \{x : |x| = 1\}$, $P_q(x)$ be a homogeneous harmonic polynomial of degree q,*

$$\delta(R) = \max_{x\in S_1}\Big\{P_q(x) - \int\limits_{|a|\le R} b_q(x,a)d\mu(a)\Big\}, \quad \delta = \varlimsup_{R\to+\infty}\delta(R)/V(R),$$

$$\overline{\Delta} = \varlimsup_{r\to+\infty} n(r,v)/r^{\rho(r)+m-2}, \quad \beta = \max(\delta,\overline{\Delta}), \quad \sigma = \varlimsup_{r\to+\infty} T(r,v)/V(r).$$

Then $\sigma = 0$, $0 < \sigma < +\infty$, $\sigma = +\infty$ if and only if, $\beta = 0$, $0 < \beta < +\infty$, $\beta = +\infty$, respectively.

We prove the following theorem.

Theorem 1. *Let u be a δ-subharmonic function in $\mathbb{R}^m$, $m \geq 2$, of finite proximate order $\rho(r)$, $\varrho(r) \to \varrho > 0$ as $r \to +\infty$. Then*

$$u(x) = -I(x, u_1) + I(x, u_2) + O\left(V(r)\right) \quad (r \to +\infty). \tag{1}$$

Proof. We note that it suffices to prove the theorem for the case of a subharmonic function.

Let v be a subharmonic function in $\mathbb{R}^m$ of proximate order $\rho(r)$, $\rho(r) \to \rho > 0$ as $r \to +\infty$, $q = [\rho]$. By Theorem 4.2 from [4] we have

$$v(x) = \int\limits_{|a|\leq 2r} K_{q-1}(x,a)d\mu(a) + \int\limits_{|a|>2r} K_q(x,a)d\mu(a) + P_q(x) + w_{q-1}(x) - \int\limits_{|a|\leq 2r} b_q(x,a)d\mu(a) =$$

$$= I_1 + I_2 + P_q(x) + w_{q-1}(x) - \int\limits_{|a|\leq 2r} b_q(x,a)d\mu(a),$$

where $P_q(x)$ is a homogeneous harmonic polynomial of degree q, $w_{q-1}(x)$ is a harmonic polynomial of degree at most $q-1$, and thus $w_{q-1}(x) = O\left(V(r)\right)$ as $r \to +\infty$.

By Lemma 4.2 from [4] (Let us remark that there is a misprint in the lemma statement. As could be seen from the proof in (4.1.4) ρ^{q+1} must be set instead of ρ^{q+2}) we obtain ($q \geq 1$, $m \geq 2$, A is some constant)

$$|I_2| \leq \left| \int\limits_{|a|>2r} K_q(x,a)d\mu(a) \right| \leq Ar^{q+1} \int\limits_{2r}^{+\infty} \frac{dn(t,v)}{t^{m+q-1}} \leq$$

$$\leq Ar^{q+1}(m+q-1) \int\limits_{2r}^{+\infty} \frac{n(t,v)}{t^{m+q}} dt = O\left(V(r)\right) \quad (r \to +\infty).$$

In the case of $q = 0$, $m = 2$ we get

$$|I_2| \leq \left| \int\limits_{|a|>2r} K_0(x,a)d\mu(a) \right| \leq -\int\limits_{2r}^{+\infty} \log\left(1 - \frac{r}{t}\right) dn(t,v) \leq$$

$$\leq \log\left(1 - \frac{r}{t}\right) n(t,v) \Big|_{+\infty}^{2r} + r\int\limits_{2r}^{+\infty} \frac{n(t,v)}{t(t-r)} dt \leq 2r \int\limits_{2r}^{+\infty} n(t,v) t^{-2} dt = O\left(V(r)\right) \quad (r \to +\infty),$$

since

$$n(r,v) = O\left(V(r)\right) = o(r), \quad r\int\limits_{r}^{+\infty} t^{\rho(t)-2} dt = \frac{1+o(1)}{1-\rho} r^{\rho(r)} \quad (r \to +\infty).$$

Now we shall estimate I_1. Let $D(x,a) = \{a : |x-a| > r, |a| \leq 2r\}$. Since

$$0 \leq \int\limits_{D(x,a)} \ln\frac{|x-a|}{r} d\mu(a) \leq n(2r,v)\ln 3, \quad \int\limits_{|a|\leq 2r} \ln\frac{r}{|a|} d\mu(a) = N(2r,v) - n(2r,v)\ln 2,$$

we have for $m = 2$

$$I_1 = \int\limits_{|a|\le 2r} K_0(x,a)d\mu(a) + O\left(V(r)\right) = -I(x,v) + \int\limits_{|a|\le 2r} \ln\frac{|x-a|}{|a|}d\mu(a) + \int\limits_{|x-a|\le r} \ln\frac{r}{|x-a|}d\mu(a) +$$

$$+O\left(V(r)\right) = -I(x,v) + \int\limits_{D(x,a)} \ln\frac{|x-a|}{r}d\mu(a) + \int\limits_{|a|\le 2r} \ln\frac{r}{|a|}d\mu(a) + O\left(V(r)\right) =$$

$$= -I(x,v) + O\left(V(r)\right) \quad (r\to+\infty).$$

In the case of $m \ge 3$ similarly to the previous

$$I_1 = \int\limits_{|a|\le 2r} K_0(x,a)d\mu(a) + O\left(V(r)\right) = -I(x,v) + \int\limits_{|x-a|\le r} \left(|x-a|^{2-m} - r^{2-m}\right)d\mu(a) +$$

$$+ \int\limits_{|a|\le 2r} \left(|a|^{2-m} - |x-a|^{2-m}\right)d\mu(a) + O\left(V(r)\right) = -I(x,v) - \int\limits_{D(x,a)} |x-a|^{2-m}d\mu(a) -$$

$$-r^{2-m}n(r,x,v) + \int\limits_{|a|\le 2r} |a|^{2-m}d\mu(a) + O\left(V(r)\right) = -I(x,v) + O\left(V(r)\right) \quad (r\to+\infty),$$

as $0 \le \int\limits_{D(x,a)} |x-a|^{2-m}d\mu(a) \le r^{2-m}n(2r,v)$, $\int\limits_{|a|\le 2r} |a|^{2-m}d\mu(a) = N(2r,v) + (2r)^{2-m}n(2r,v)$. Thus, we obtain

$$v(x) = -I(x,v) + P_q(x) - \int\limits_{|a|\le 2r} b_q(x,a)d\mu(a) + O\left(V(r)\right) \quad (r\to+\infty).$$

In the case of non-integer order of the subharmonic function v, that $P_q(x) = O\left(V(r)\right)$ and by Lemma 4.1 from [4]

$$\int\limits_{|a|\le 2r} b_q(x,a)d\mu(a) = O\left(V(r)\right) \quad (r\to+\infty).$$

Hence

$$v(x) = -I(x,v) + O\left(V(r)\right) \quad (r\to+\infty).$$

Taking into account that $\rho(r)$ is a proximate order of the function v, which means $0 < \sigma < +\infty$, by theorem A in case of integer order of v we obtain

$$P_q(x) - \int\limits_{|a|\le 2r} b_q(x,a)d\mu(a) = O\left(V(r)\right) \quad (r\to+\infty),$$

whence $v(x) = -I(x,v) + O\left(V(r)\right)$ as $r\to+\infty$. □

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