ASYMPTOTICS OF $\delta$-SUBHARMONIC FUNCTIONS OF FINITE ORDER


For a $\delta$-subharmonic in $\mathbb{R}^m$, $m \geq 2$, function $u = u_1 - u_2$ of finite positive order we found the asymptotical representation of the form

$$ u(x) = -I(x, u_1) + I(x, u_2) + O(V(|x|)), \ x \to \infty, $$

where

$$ I(x, u_i) = \int_{|a-x| \leq |x|} K(x, a) d\mu_i(a), \ K(x, a) = \ln \frac{|x|}{|x-a|} $$

for $m = 2$, $K(x, a) = |x-a|^{2-m}-|x|^{2-m}$ for $m \geq 3$, $\mu_i$ is the Riesz measure of the subharmonic function $u_i$, $V(r) = r^{\rho(r)}$, $\rho(r)$ is a proximate order of $u$. The obtained result generalizes one theorem of I. F. Krasichkov for entire functions.

In this paper one theorem of I.F. Krasichkov (6, theorem 1) is generalized to the case of $\delta$-subharmonic in $\mathbb{R}^m$, $m \geq 2$, functions of finite positive order.

Let $u$ be a $\delta$-subharmonic function in $\mathbb{R}^m$, $m \geq 2$, that is $u = u_1 - u_2$, where $u_1$, $u_2$ are subharmonic functions. Without loss of generality, we assume that $u_1$, $u_2$ are harmonic in a neighborhood of zero and their orders do not exceed the order of $u$, the Riesz masses of $u_1$ and $u_2$ are concentrated on disjoint sets and $u_1(0) = u_2(0) = 0$. We set

$$ M(a, r) = \ln(r/|a|), \ L(x, a) = \ln (|x|/|x-a|) $$

for $m = 2$ and

$$ M(a, r) = |a|^{2-m}-r^{2-m}, \ L(x, a) = |x-a|^{2-m}-|x|^{2-m} $$

for $m \geq 3$, $x, a \in \mathbb{R}^m$, $r \geq 0$. Let $\mu_i$ be a Riesz measure of subharmonic function $u_i$, $|x| = r$, $n(t, u_i) = \mu_i (\{h : |h| \leq t\})$, $n(t, x, u_i) = \mu_i (\{h : |h-x| \leq t\})$,

$d_m = m - 2$, for $m \geq 3$, $d_2 = 1$. We put

$$ N(r, u_i) = \int_{|a| \leq r} M(a, r) d\mu_i(a) = d_m \int_0^r n(t, u_i) t^{1-m} dt, $$

$$ I(x, u_i) = \int_{|x-a| \leq r} L(x, a) d\mu_i(a) = d_m \int_0^r n(t, x, u_i) t^{1-m} dt. $$

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The quantity $I(x,u_i)$ is called a concentration index of a subharmonic function $u_i$ at the point $x$ [3, p. 227]. The similar term was used for a related concept in case of entire functions by I.F. Krasichkov [6]. In fact, the quantity $I(x,u_i)$ was introduced in the book of B.Ya. Levin [7, p. 164] (see also [5]) before [6].

Let $\rho(r)$ be a proximate order of the function $u$, that is:

1) $\rho(r) \geq 0$ on $[0, +\infty)$;
2) $\rho(r) \to \rho > 0$ as $r \to +\infty$;
3) the function $\rho(r)$ is continuously differentiable on $[0, +\infty)$;
4) $r\rho'(r)\ln r \to 0$ as $r \to +\infty$;
5) $0 < \lim_{r \to +\infty} T(r, u)/V(r) < +\infty$, with $V(r) = r^\rho(r)$,

where $T(r, u)$ is the Nevanlinna characteristic of the function $u$, i.e.

$$
T(r, u) = \frac{1}{|S^{m-1}|} \int_{S^{m-1}} \max\{u_1(rx), u_2(rx)\}dS(x), \quad 0 \leq r < +\infty,
$$

$dS(x)$ is the sphere area element of $S^{m-1} = \{x \in \mathbb{R}^m : |x| = 1\}$, and $|S^{m-1}| = 2\pi^{m/2}/\Gamma(m/2)$. If $u$ is a subharmonic function on $\mathbb{R}^m (m \geq 2)$, $u(0) = 0$, then

$$
T(r, u) = \frac{1}{|S^{m-1}|} \int_{S^{m-1}} u^+(rx)dS(x), \quad u^+ = \max\{u, 0\}, \quad 0 \leq r < +\infty.
$$

We set $K(x) = \log |x|$ for $m = 2$ and $K(x) = -|x|^{2-m}$ for $m \geq 3$. The function $K(x-a)$ is harmonic in $\mathbb{R}^m$ except the point $x = a$. In particular, if $a \neq 0$ then $K(x-a)$ can be represented as a power series in variables $x_1, x_2, \ldots, x_m$, convergent in the neighborhood of the origin. We have

$$
K(x-a) = \sum_{\nu=0}^{+\infty} b_\nu(x,a),
$$

where for fixed $\nu$ and $a \neq 0$ by $b_\nu(x,a)$ we denote a homogeneous harmonic polynomial of $x_1, x_2, \ldots, x_m$, whose degree is $\nu$. Let us set

$$
K_q(x,a) = K(x-a) - \sum_{\nu=0}^{q} b_\nu(x,a).
$$

The Brelot’s results ([2, pp.145, 147], Theorems 1, 2) imply the analogue of Lindelöf theorem for subharmonic functions of finite order ([1, p. 9]).

**Theorem A (Analogue of the Lindelöf theorem).** Let $v$ be a subharmonic function of integer order $\rho$, $S_1 = \{x : |x| = 1\}$, $P_q(x)$ be a homogeneous harmonic polynomial of degree $q$.

$$
\delta(R) = \max_{x \in S_1} \left\{ P_q(x) - \int_{|a| \leq R} b_q(x,a)d\mu(a) \right\}, \quad \delta = \lim_{R \to +\infty} \delta(R)/V(R),
$$

$$
\Delta = \lim_{r \to +\infty} n(r,v)/r^{\rho(r)+m-2}, \quad \beta = \max(\delta, \Delta), \quad \sigma = \lim_{r \to +\infty} T(r,v)/V(r).
$$

Then $\sigma = 0$, $0 < \sigma < +\infty$, $\sigma = +\infty$ if and only if, $\beta = 0$, $0 < \beta < +\infty$, $\beta = +\infty$, respectively.
We prove the following theorem.

**Theorem 1.** Let \( u \) be a \( \delta \)-subharmonic function in \( \mathbb{R}^m \), \( m \geq 2 \), of finite proximate order \( \rho(r), g(r) \to q > 0 \) as \( r \to +\infty \). Then

\[
u(x) = -I(x, u_1) + I(x, u_2) + O(V(r)) \quad (r \to +\infty).
\]

**Proof.** We note that it suffices to prove the theorem for the case of a subharmonic function.

Let \( v \) be a subharmonic function in \( \mathbb{R}^m \) of proximate order \( \rho(r), \rho(r) \to \rho > 0 \) as \( r \to +\infty \), \( q = [\rho] \). By Theorem 4.2 from [4] (Let us remark that there is a misprint in the lemma statement. As could be seen from the proof in (4.1.4) \( \rho^{q+1} \) must be set instead of \( \rho^{q+2} \)) we obtain \( (q \geq 1, m \geq 2, A \) is some constant)

\[
|J_2| \leq \int_{|a|>2r} K_q(x,a)\,d\mu(a) \leq Ar^{q+1} \int_{2r}^{+\infty} \frac{dn(t,v)}{t^{m+q-1}} \leq Ar^{q+1}(m+q-1) \int_{2r}^{+\infty} \frac{n(t,v)}{t^{m+q}} \,dt = O(V(r)) \quad (r \to +\infty).
\]

In the case of \( q = 0, m = 2 \) we get

\[
|J_2| \leq \int_{|a|>2r} K_0(x,a)\,d\mu(a) \leq -\int_{2r}^{+\infty}\log\left(1 - \frac{r}{t}\right) \,dn(t,v) \leq \log\left(1 - \frac{r}{t}\right) n(t,v) + r \int_{2r}^{+\infty} \frac{n(t,v)}{t(t-r)} \,dt \leq 2r \int_{2r}^{+\infty} n(t,v) t^2 \,dt = O(V(r)) \quad (r \to +\infty),
\]

since

\[
n(r,v) = O(V(r)) = o(r), \quad r \int \rho(t)^{-2} \,dt = \frac{1 + o(1)}{1 - \rho} \quad (r \to +\infty).
\]

Now we shall estimate \( I_1 \). Let \( D(x,a) = \{a : |x-a| > r, |a| \leq 2r \} \). Since

\[
0 \leq \int_{D(x,a)} \ln \frac{|x-a|}{r} \,d\mu(a) \leq n(2r,v) \ln 3, \quad \int_{|a| \leq 2r} \ln \frac{r}{|a|} \,d\mu(a) = N(2r,v) - n(2r,v) \ln 2,
\]
we have for $m = 2$

$$I_1 = \int_{|a| \leq 2r} K_0(x, a) d\mu(a) + O(V(r)) = -I(x, v) + \int_{|a| \leq 2r} \ln \frac{|x-a|}{|a|} d\mu(a) + \int_{|x-a| \leq r} \ln \frac{r}{|x-a|} d\mu(a) + O(V(r)) = -I(x, v) + O(V(r)) \quad (r \to +\infty).$$

In the case of $m \geq 3$ similarly to the previous

$$I_1 = \int_{|a| \leq 2r} K_0(x, a) d\mu(a) + O(V(r)) = -I(x, v) + \int_{|a| \leq 2r} (|x-a|^{2-m} - r^{2-m}) d\mu(a) +$$

$$+ \int_{|a| \leq 2r} (|a|^{2-m} - |x-a|^{2-m}) d\mu(a) + O(V(r)) = -I(x, v) - \int_{D(x,a)} |x-a|^{2-m} d\mu(a) -$$

$$-r^{2-m} n(r, x, v) + \int_{|a| \leq 2r} |a|^{2-m} d\mu(a) + O(V(r)) = -I(x, v) + O(V(r)) \quad (r \to +\infty),$$

as $0 \leq \int_{D(x,a)} |x-a|^{2-m} d\mu(a) \leq r^{2-m} n(2r, v), \quad \int_{|a| \leq 2r} |a|^{2-m} d\mu(a) = N(2r, v) + (2r)^{2-m} n(2r, v)$. Thus, we obtain

$$v(x) = -I(x, v) + P_q(x) - \int_{|a| \leq 2r} b_q(x, a) d\mu(a) + O(V(r)) \quad (r \to +\infty).$$

In the case of non-integer order of the subharmonic function $v$, that $P_q(x) = O(V(r))$ and by Lemma 4.1 from [4]

$$\int_{|a| \leq 2r} b_q(x, a) d\mu(a) = O(V(r)) \quad (r \to +\infty).$$

Hence

$$v(x) = -I(x, v) + O(V(r)) \quad (r \to +\infty).$$

Taking into account that $\rho(r)$ is a proximate order of the function $v$, which means $0 < \sigma \leq +\infty$, by theorem A in case of integer order of $v$ we obtain

$$P_q(x) - \int_{|a| \leq 2r} b_q(x, a) d\mu(a) = O(V(r)) \quad (r \to +\infty),$$

whence $v(x) = -I(x, v) + O(V(r))$ as $r \to +\infty$. \qed
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