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T. Biswas, C. Biswas

GENERALIZED (α, β) ORDER BASED ON SOME GROWTH PROPERTIES OF WRONSKIANS

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In this paper the comparative growth properties of composition of entire and meromorphic functions on the basis of their generalized (α, β) order and generalized lower (α, β) order of Wronskians generated by entire and meromorphic functions have been investigated.

1. Introduction, definitions and notations. Let us consider that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna theory of meromorphic functions which are available in [4, 5, 10]. We also use the standard notations and definitions of the theory of entire functions which are available in [9] and therefore we do not explain those in details. Let f be an entire function defined in the open complex plane $\mathbb{C}$ and $M_f(r) = \max\{|f(z)| : |z| = r\}$. When f is meromorphic, one may introduce another function $T_f(r)$ known as Nevanlinna's characteristic function of f (see [4, p.4]), playing the same role as $M_f(r)$. The Nevanlinna's characteristic function of a meromorphic function f is defined as $T_f(r) = N_f(r)+m_f(r)$, wherever the function $N_f(r, a)$ known as counting function of a-points of meromorphic f is defined as follows:

$$N_f(r, a) = \int_0^r \frac{n_f(t, a) - n_f(0, a)}{t} dt + n_f(0, a) \log r,$$

in addition we represent by $n_f(r, a)$ the number of a-points of f in $|z| \le r$ and an ∞ -point is a pole of f. In many occasions $N_f(r, \infty)$ is symbolized by $N_f(r)$.

On the other hand, the function $m_f(r, \infty)$ alternatively indicated by $m_f(r)$ known as the proximity function of f is defined as $m_f(r) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})|d\theta$, where $\log^+ x = \max(\log x, 0)$ for all $x \geqslant 0$. Also we may employ $m(r, \frac{1}{f-a})$ by $m_f(r, a)$. If f is entire, then the Nevanlinna's characteristic function $T_f(r)$ of f is defined as $T_f(r) = m_f(r)$.

For a meromorphic function f defined on $\mathbb{C}$, the Wronskian determinant $W(f)$ is defined

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as

$$W(f) = W(a_1, a_2, \ldots, a_k, f) = \begin{vmatrix} a_1 & a_2 & . & . & . & a_k & f \\ a_1^{'} & a_2^{'} & . & . & . & a_k^{'} & f^{'} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ a_1^{(k)} & a_2^{(k)} & . & . & . & a_k^{(k)} & f^{(k)} \end{vmatrix}$$

where a_1, a_2, ... a_k are linearly independent meromorphic functions and small with respect to f i.e., $T_{a_i}(r) = S_f(r)$ for $i = 1, 2, 3 \ldots k$. From the Nevanlinna's second fundamental theorem, it follows that the set of values of $a \in \mathbb{C} \cup \{\infty\}$ for which $\delta(a; f) > 0$ is countable and $\sum\limits_{a \neq \infty} \delta(a; f) + \delta(\infty; f) \leq 2$ (cf [4], p.43) where $\delta(a; f) = 1 - \limsup\limits_{r\to\infty} \frac{N(r,a;f)}{T_f(r)} = \liminf\limits_{r\to\infty} \frac{m(r,a;f)}{T_f(r)}$. If in particular $\sum\limits_{a \neq \infty} \delta(a; f) + \delta(\infty; f) = 2$, we say that f has the maximum deficiency sum.

However, let L be a class of continuous non-negative on $(-\infty, +\infty)$ function α such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$ with $\alpha(x) \uparrow +\infty$ as $x \to +\infty$. For any $\alpha \in L$, we say that $\alpha \in L_1^0$, if $\alpha((1 + o(1))x) = (1 + o(1))\alpha(x)$ as $x \to +\infty$ and $\alpha \in L_2^0$, if $\alpha(\exp((1 + o(1))x)) = (1 + o(1))\alpha(\exp(x))$ as $x \to +\infty$. Finally for any $\alpha \in L$, we also say that $\alpha \in L_1$, if $\alpha(cx) = (1 + o(1))\alpha(x)$ as $x_0 \leq x \to +\infty$ for each $c \in (0, +\infty)$ and $\alpha \in L_2$, if $\alpha(\exp(cx)) = (1 + o(1))\alpha(\exp(x))$ as $x_0 \leq x \to +\infty$ for each $c \in (0, +\infty)$. Clearly, $L_1 \subset L_1^0$, $L_2 \subset L_2^0$ and $L_2 \subset L_1$.

Considering this, the value

$$\varrho_{(\alpha,\beta)}[f] = \overline{\lim_{r\to+\infty}} \frac{\alpha(\log M_f(r))}{\beta(\log r)} \quad (\alpha \in L, \beta \in L)$$

is called [7] *generalized* (α, β) *order* of an entire function f. For details about generalized order (α, β) one may see [7]. During the past decades, several authors made close investigations on the properties of entire functions related to generalized order (α, β) in some different direction. For the purpose of further applications, here in this paper we rewrite the definition of the generalized (α, β) order of entire and meromorphic function in the following way after giving a minor modification to the original definition (e.g. see, [7]) which considerably extend the definition of φ-order of entire and meromorphic function introduced in [3].

Definition 1. Let $\alpha, \beta \in L$. The generalized order (α, β) and generalized lower (α, β) order of a meromorphic function f are defined as $\begin{matrix} \varrho_{(\alpha,\beta)}[f] \\ \lambda_{(\alpha,\beta)}[f] \end{matrix} = \lim\limits_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \dfrac{\alpha(\exp(T_f(r)))}{\beta(r)}$, If f is an entire function, then $\begin{matrix} \varrho_{(\alpha,\beta)}[f] \\ \lambda_{(\alpha,\beta)}[f] \end{matrix} = \lim\limits_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \dfrac{\alpha(M_f(r))}{\beta(r)}$.

Using the inequality $T_f(r) \leq \log M_f(r) \leq 3T_f(2r)$ (see [4]), for an entire function f, one may easily verify that

$$\begin{matrix} \varrho_{(\alpha,\beta)}[f] \\ \lambda_{(\alpha,\beta)}[f] \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\alpha(M_f(r))}{\beta(r)} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\alpha(\exp(T_f(r)))}{\beta(r)},$$

when $\alpha \in L_2$ and $\beta \in L_1$.

In this paper we wish to prove some newly developed results relating to the growth properties of composite entire and meromorphic functions on the basis of generalized (α, β) order and generalized lower order (α, β) of Wronskians generated by entire and meromorphic functions.

2. Lemmas. In this section we present some lemmas which will be needed in the sequel.

Lemma 1 ([1]). *If f is a meromorphic function and g is an entire function then as $r \to +\infty$*
$$T_{f(g)}(r) \leqslant (1+o(1))\frac{T_g(r)}{\log M_g(r)} T_f(M_g(r)).$$

Lemma 2 ([2]). *Let f and g are any two entire functions with $g(0) = 0$. Also let β satisfy $0 < \beta < 1$ and $c(\beta) = \frac{(1-\beta)^2}{4\beta}$. Then for all sufficiently large values of r,*
$$M_f(c(\beta)M_g(\beta r)) \leq M_{f(g)}(r).$$
In addition if $\beta = \frac{1}{2}$, then for all sufficiently large values of r, $M_{f(g)}(r) \geq M_f(\dfrac{1}{8} M_g(\dfrac{r}{2}))$.

Lemma 3 ([6]). *Let f be a transcendental meromorphic function having the maximum deficiency sum. Then*
$$\lim_{r\to+\infty} \frac{T_{W(f)}(r)}{T_f(r)} = 1 + k - k\delta(\infty; f).$$

Lemma 4. *Let f be a transcendental meromorphic function having the maximum deficiency sum. Then the generalized (α, β) order and generalized lower (α, β) order of $W(f)$ and that of f are same where $\alpha \in L_2$.*

Proof. Since $\alpha \in L_2$, from Lemma 3 we get $\alpha(\exp(T_{W(f)}(r))) \sim \alpha(\exp(T_f(r)))$ as $r \to +\infty$ and, thus, $\varrho_{(\alpha,\beta)}[W(f)] = \varrho_{(\alpha,\beta)}[f]$ and $\lambda_{(\alpha,\beta)}[W(f)] = \lambda_{(\alpha,\beta)}[f]$. □

3. Main results. In this section we present the main results of the paper. Below we suppose that functions α_2, β_1, β_2, β_3 belong to the class L_1 and α_1, α_3 belong to the class L_2 unless otherwise specifically stated.

Theorem 1. *Let f be a transcendental meromorphic function having the maximum deficiency and g be an entire function such that $0 < \lambda_{(\alpha_1,\beta_1)}[f] \leq \varrho_{(\alpha_1,\beta_1)}[f] < +\infty$. and $\varrho_{(\alpha_2,\beta_2)}[g] < +\infty$. Also let γ be a positive continuous on $[0, +\infty)$ function increasing to $+\infty$ and $A \geq 0$ be any number. If $\beta_1(\alpha_2^{-1}(\log r)) \leq r$ and $\lim\limits_{r\to+\infty} \frac{\log \gamma(r)}{\log r} = +\infty$, then*
$$\lim_{r\to+\infty} \frac{\{\alpha_1(\exp(T_{f(g)}(\beta_2^{-1}(\log r))))\}^{1+A}}{\alpha_1(\exp(T_{W(f)}(\beta_1^{-1}(\gamma(r)))))} = 0. \tag{1}$$
If either $\beta_1(r) = B\alpha_2(r)$, where $B =$ constant > 0 and $\lim\limits_{r\to+\infty} \frac{\log\gamma(r)}{\log r} = +\infty$ or $\beta_1(\alpha_2^{-1}(r)) \in L_1$ and $\lim\limits_{r\to+\infty} \frac{\log\gamma(r)}{\beta_1(\alpha_2^{-1}(\log r))} = +\infty$, then
$$\lim_{r\to+\infty} \frac{\{\exp(\alpha_1(\exp(T_{f(g)}(\beta_2^{-1}(\log r)))))\}^{1+A}}{\alpha_1(\exp(T_{W(f)}(\beta_1^{-1}(\gamma(r)))))} = 0. \tag{2}$$

Proof. From the definition of $\lambda_{(\alpha_1,\beta_1)}[W(f)]$ and in view of Lemma 4, we get
$$\alpha_1(\exp(T_{W(f)}(\beta_1^{-1}(\gamma(r))))) \geq (\lambda_{(\alpha_1,\beta_1)}[W(f)] - \varepsilon)\gamma(r) = (\lambda_{(\alpha_1,\beta_1)}[f] - \varepsilon)\gamma(r) \tag{3}$$
for every $\varepsilon \in (0, \lambda_{(\alpha_1,\beta_1)}[f])$ and all $r \geq r_0(\varepsilon)$. On the other hand, in view of Lemma 1 and the inequality $T_g(r) \leq \log^+ M_g(r)$ we get
$$\alpha_1(\exp(T_{f(g)}(\beta_2^{-1}(\log r)))) \leqslant (1+o(1))(\varrho_{(\alpha_1,\beta_1)}[f] + \varepsilon)\beta_1(M_g(\beta_2^{-1}(\log r))) \tag{4}$$

for every $\varepsilon > 0$ and all $r \geq r_0(\varepsilon)$.

If $\beta_1(\alpha_2^{-1}(\log r)) \leq r$ then (4) implies

$$\alpha_1(\exp(T_{f(g)}(\beta_2^{-1}(\log r)))) \leqslant (1+o(1))(\varrho_{(\alpha_1,\beta_1)}[f]+\varepsilon)\beta_1(\alpha_2^{-1}(\alpha_2(M_g(\beta_2^{-1}(\log r)))))$$
$$\leqslant (1+o(1))(\varrho_{(\alpha_1,\beta_1)}[f]+\varepsilon)\beta_1(\alpha_2^{-1}(\log r^{(\varrho_{(\alpha_2,\beta_2)}[g]+\varepsilon)})) \leq$$
$$\leqslant (1+o(1))(\varrho_{(\alpha_1,\beta_1)}[f]+\varepsilon)r^{(\varrho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}. \tag{5}$$

If $\lim\limits_{r\to+\infty}\frac{\log\gamma(r)}{\log r} = +\infty$, then from (3) and (5) we get

$$\frac{\{\alpha_1(\exp(T_{f(g)}(\beta_2^{-1}(\log r)))\}^{1+A}}{\alpha_1(\exp(T_{W(f)}(\beta_1^{-1}(\gamma(r)))))} \leq$$
$$\leqslant \frac{(1+o(1))(\varrho_{(\alpha_1,\beta_1)}[f]+\varepsilon)^{1+A}[(\varrho_{(\alpha_1,\beta_1)}[f]+\varepsilon)r^{(\varrho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}]^{1+A}}{(\lambda_{(\alpha_1,\beta_1)}[f]-\varepsilon)\gamma(r)} = o(1)$$

as $r \to +\infty$, i.e. (1) is proved.

If $\beta_1(r) = B\alpha_2(r)$ then from (4) as above we have

$$\alpha_1(\exp(T_{f(g)}(\beta_2^{-1}(\log r)))) \leqslant (1+o(1))B(\varrho_{(\alpha_1,\beta_1)}[f]+\varepsilon)\alpha_2(M_g(\beta_2^{-1}(\log r))) \leq$$
$$\leq (1+o(1))B(\varrho_{(\alpha_1,\beta_1)}[f]+\varepsilon)(\varrho_{(\alpha_2,\beta_2)}[g]+\varepsilon)\log r,$$

i.e., $\exp(\alpha_1(\exp(T_{f(g)}(\beta_2^{-1}(\log r))))) \leqslant (1+o(1))r^{B(\varrho_{(\alpha_1,\beta_1)}[f]+\varepsilon)(\varrho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}$. Hence in view of (3) and the condition $\lim\limits_{r\to+\infty}\frac{\log\gamma(r)}{\log r} = +\infty$ we get

$$\frac{\{\exp(\alpha_1(\exp(T_{f(g)}(\beta_2^{-1}(\log r)))))\}^{1+A}}{\alpha_1(\exp(T_{W(f)}(\beta_1^{-1}(\gamma(r)))))} \leqslant \frac{(1+o(1))r^{B(\varrho_{(\alpha_1,\beta_1)}[f]+\varepsilon)(\varrho_{(\alpha_2,\beta_2)}[g]+\varepsilon)(1+A)}}{(\lambda_{(\alpha_1,\beta_1)}[f]-\varepsilon)\gamma(r)} = o(1)$$

as $r \to +\infty$, i.e. (2) is proved.

Finally if $\beta_1(\alpha_2^{-1}(r)) \in L_1$ then as above we have

$$\alpha_1(\exp(T_{f(g)}(\beta_2^{-1}(\log r)))) \leqslant (1+o(1))(\varrho_{(\alpha_1,\beta_1)}[f]+\varepsilon)\beta_1(\alpha_2^{-1}(\log r)),$$

i.e.,

$$\exp(\alpha_1(\exp(T_{f(g)}(\beta_2^{-1}(\log r))))) \leqslant \exp((1+o(1))(\varrho_{(\alpha_1,\beta_1)}[f]+\varepsilon)\beta_1(\alpha_2^{-1}(\log r)))$$

whence in view of (3) and the condition $\lim\limits_{r\to+\infty}\frac{\log\gamma(r)}{\beta_1(\alpha_2^{-1}(\log r))} = +\infty$ we get

$$\frac{\{\exp(\alpha_1(\exp(T_{f(g)}(\beta_2^{-1}(\log r)))))\}^{1+A}}{\alpha_1(\exp(T_{W(f)}(\beta_1^{-1}(\gamma(r)))))} \leq$$
$$\leqslant \frac{[\exp((1+o(1))(\varrho_{(\alpha_1,\beta_1)}[f]+\varepsilon)\beta_1(\alpha_2^{-1}(\log r)))]^{1+A}}{(\lambda_{(\alpha_1,\beta_1)}[f]-\varepsilon)\gamma(r)} = o(1)$$

as $r \to +\infty$, i.e. (2) is proved again. The proof of Theorem 1 is completed. □

Remark 1. Theorem 1 improves and extends Theorem 3 of [8].

Remark 2. In Theorem 1 if we take the condition "$\varrho_{(\alpha_1,\beta_1)}[f] > 0$" instead of "$0 < \lambda_{(\alpha_1,\beta_1)}[f] \leq \varrho_{(\alpha_1,\beta_1)}[f] < +\infty$", the theorem remains true with " limit inferior" in place of "limit ".

Remark 3. In Theorem 1 the conditions "$\sum_{a\neq\infty}\delta(a;f)+\delta(\infty;f)=2$", "$0<\lambda_{(\alpha_1,\beta_1)}[f]\leq\varrho_{(\alpha_1,\beta_1)}[f]<+\infty$" and "$\varrho_{(\alpha_2,\beta_2)}[g]<+\infty$" can be replaced by the conditions "$\sum_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2$", "$\varrho_{(\alpha_1,\beta_1)}[f]<+\infty$" and "$0<\lambda_{(\alpha_2,\beta_2)}[g]\leq\varrho_{(\alpha_2,\beta_2)}[g]<+\infty$" ($\alpha_2\in L_2$) Then the conclusion of Theorem 1 remains true with "$\alpha_1(\exp(T_{W(f)}(\beta_1^{-1}(\gamma(r)))))$" replaced by "$\alpha_2(\exp(T_{W(g)}(\beta_2^{-1}(\gamma(r)))))$".

Remark 4. In Remark 3, if we take the condition "$\varrho_{(\alpha_2,\beta_2)}[g]>0$" instead of "$0<\lambda_{(\alpha_2,\beta_2)}[g]\leq\varrho_{(\alpha_2,\beta_2)}[g]<+\infty$", the theorem remains true with "limit replaced by limit inferior".

Theorem 2. *Let f be a transcendental meromorphic function with $\sum_{a\neq\infty}\delta(a;f)+\delta(\infty;f)=2$ and g be an entire function such that $0<\lambda_{(\alpha_1,\beta_1)}[f]\leq\varrho_{(\alpha_1,\beta_1)}[f]<+\infty$. and $\varrho_{(\alpha_2,\beta_2)}[g]<+\infty$. If $\alpha_2(\beta_1^{-1}(r))\in L_1$, then*

$$\overline{\lim_{r\to+\infty}}\frac{\alpha_2(\beta_1^{-1}(\alpha_1(\exp(T_{f(g)}(r)))))}{\alpha_1(\exp(T_{W(f)}(\beta_1^{-1}(\beta_2(r)))))}\leq\frac{\varrho_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_1,\beta_1)}[f]}.$$

Proof. In view of (3), it follows for all sufficiently large values of r that

$$\alpha_1(\exp(T_{W(f)}(\beta_1^{-1}(\beta_2(r)))))\geq(\lambda_{(\alpha_1,\beta_1)}[f]-\varepsilon)\beta_2(r). \tag{6}$$

Again in view of (4), we get that $\alpha_1(\exp(T_{f(g)}(r)))\leqslant(1+o(1))(\varrho_{(\alpha_1,\beta_1)}[f]+\varepsilon)\beta_1(M_g(r))$ as $r\to+\infty$. Since $\alpha_2(\beta_1^{-1}(r))\in L_1$, we obtain from above for all sufficiently large values of r that $\alpha_2(\beta_1^{-1}(\alpha_1(\exp(T_{f(g)}(r)))))\leq(1+o(1))\alpha_2(M_g(r))$, i.e.,

$$\alpha_2(\beta_1^{-1}(\alpha_1(\exp(T_{f(g)}(r)))))\leq(1+o(1))(\varrho_{(\alpha_2,\beta_2)}[g]+\varepsilon)\beta_2(r). \tag{7}$$

Now combining (6) and above we get that $\overline{\lim_{r\to+\infty}}\frac{\alpha_2(\beta_1^{-1}(\alpha_1(\exp(T_{f(g)}(r)))))}{\alpha_1(\exp(T_{W(f)}(\beta_1^{-1}(\beta_2(r)))))}\leq\frac{\varrho_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_1,\beta_1)}[f]}$. Hence the theorem follows. □

Remark 5. We remark that in Theorem 2 if we will replace the condition $\varrho_{(\alpha_2,\beta_2)}[g]<+\infty$ by $\lambda_{(\alpha_2,\beta_2)}[g]<+\infty$, then

$$\underline{\lim_{r\to+\infty}}\frac{\alpha_2(\beta_1^{-1}(\alpha_1(\exp(T_{f(g)}(r)))))}{\alpha_1(\exp(T_{W(f)}(\beta_1^{-1}(\beta_2(r))))}\leq\frac{\lambda_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_1,\beta_1)}[f]}. \tag{8}$$

If the conditions of Theorem 2 remain unchanged then (8) remains true with "$\lambda_{(\alpha_2,\beta_2)}[g]$" and "$\lambda_{(\alpha_1,\beta_1)}[f]$" replaced by "$\varrho_{(\alpha_2,\beta_2)}[g]$" and "$\varrho_{(\alpha_1,\beta_1)}[f]$" respectively.

Remark 6. In Theorem 2 the conditions "$\sum_{a\neq\infty}\delta(a;f)+\delta(\infty;f)=2$", "$0<\lambda_{(\alpha_1,\beta_1)}[f]\leq\varrho_{(\alpha_1,\beta_1)}[f]<+\infty$" and "$\varrho_{(\alpha_2,\beta_2)}[g]<+\infty$" can be replaced by the conditions "$\sum_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2$", "$0<\varrho_{(\alpha_1,\beta_1)}[f]<+\infty$" and "$0<\lambda_{(\alpha_2,\beta_2)}[g]\leq\varrho_{(\alpha_2,\beta_2)}[g]<+\infty$" ($\alpha_2\in L_2$) Then the conclusion of Theorem 1 remains true with "$\alpha_1(\exp(T_{W(f)}(\beta_1^{-1}(\beta_2(r))))$" and "$\lambda_{(\alpha_1,\beta_1)}[f]$" replaced by "$\alpha_2(\exp(T_{W(g)}(r)))$" and "$\lambda_{(\alpha_2,\beta_2)}[g]$" respectively.

Remark 7. In Remark 6 if we take the condition "$0<\lambda_{(\alpha_2,\beta_2)}[g]<+\infty$" or "$0<\varrho_{(\alpha_2,\beta_2)}[g]<+\infty$" instead of "$0<\lambda_{(\alpha_2,\beta_2)}[g]\leq\varrho_{(\alpha_2,\beta_2)}[g]<+\infty$", then $\underline{\lim_{r\to+\infty}}\frac{\alpha_2(\beta_1^{-1}(\alpha_1(\exp(T_{f(g)}(r)))))}{\alpha_2(\exp(T_{W(g)}(r)))}\leq1$.

Theorem 3. *Let f be a transcendental meromorphic function with $\sum_{a\neq\infty}\delta(a;f)+\delta(\infty;f)=2$ and g be a transcendental entire function having the maximum deficiency sum such that $\varrho_{(\alpha_1,\beta_1)}[f(g)]<\infty$ and $\lambda_{(\alpha_3,\beta_3)}[g]>0$. Then*

$$\lim_{r\to\infty}\frac{\{\alpha_1(\exp(T_{f(g)}(\beta_1^{-1}(\log r))))\}^2}{\alpha_3(\exp(T_{W(g)}(\beta_3^{-1}(\log r))))\cdot\alpha_3(\exp(T_{W(g)}(\beta_3^{-1}(r))))}=0.$$

Proof. In view of (3), we get

$$\alpha_3(\exp(T_{W(g)}(\beta_3^{-1}(\log r))))\geq(\lambda_{(\alpha_3,\beta_3)}[g]-\varepsilon)\log r \tag{9}$$

and

$$\alpha_3(\exp(T_{W(g)}(\beta_3^{-1}(r))))\geq(\lambda_{(\alpha_3,\beta_3)}[g]-\varepsilon)r \tag{10}$$

for every $\varepsilon\in(0,\lambda_{(\alpha_3,\beta_3)}[g])$ and all $r\geq r_0(\varepsilon)$.

Further for arbitrary positive ε we obtain for all sufficiently large values of r

$$\alpha_1(\exp(T_{f(g)}(\beta_1^{-1}(\log r))))\leq(\varrho_{(\alpha_1,\beta_1)}[f(g)]+\varepsilon)\log r. \tag{11}$$

Now from (9) and (11) we have for all sufficiently large values of r that

$$\frac{\alpha_1(\exp(T_{f(g)}(\beta_1^{-1}(\log r))))}{\alpha_3(\exp(T_{W(g)}(\beta_3^{-1}(\log r))))}\leq\frac{(\varrho_{(\alpha_1,\beta_1)}[f(g)]+\varepsilon)\log r}{(\lambda_{(\alpha_3,\beta_3)}[g]-\varepsilon)\log r}.$$

As $\varepsilon>0$ is arbitrary we obtain from above that

$$\overline{\lim_{r\to+\infty}}\frac{\alpha_1(\exp(T_{f(g)}(\beta_1^{-1}(\log r))))}{\alpha_3(\exp(T_{W(g)}(\beta_3^{-1}(\log r))))}\leq\frac{\varrho_{(\alpha_1,\beta_1)}[f(g)]}{\lambda_{(\alpha_3,\beta_3)}[g]}. \tag{12}$$

Again from (10) and (11) we get for all sufficiently large values of r that

$$\frac{\alpha_1(\exp(T_{f(g)}(\beta_1^{-1}(\log r))))}{\alpha_3(\exp(T_{W(g)}(\beta_3^{-1}(r))))}\leq\frac{(\varrho_{(\alpha_1,\beta_1)}[f(g)]+\varepsilon)\log r}{(\lambda_{(\alpha_3,\beta_3)}[g]-\varepsilon)r}.$$

Since $\varepsilon>0$ is arbitrary it follows from above that

$$\lim_{r\to+\infty}\frac{\alpha_1(\exp(T_{f(g)}(\beta_1^{-1}(\log r))))}{\alpha_3(\exp(T_{W(g)}(\beta_3^{-1}(r))))}=0. \tag{13}$$

Thus the theorem follows from (12) and (13). □

Theorem 4. *Let f be a transcendental meromorphic function with $\sum_{a\neq\infty}\delta(a;f)+\delta(\infty;f)=2$ and g be a transcendental entire function having the maximum deficiency sum such that $\varrho_{(\alpha_2,\beta_2)}[g]<\lambda_{(\alpha_1,\beta_1)}[f]\leq\varrho_{(\alpha_1,\beta_1)}[f]$ ($\beta_1\in L_2$). Also let C be any positive constant.*
If $\alpha_2(\beta_1^{-1}(r))\in L_1$, then

$$\lim_{r\to+\infty}\frac{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(\exp(T_{f(g)}(\beta_2^{-1}(\log r))))))\cdot\alpha_1(\exp(T_{f(g)}(\beta_2^{-1}(\log r))))}{\exp(\alpha_1(\exp(T_{W(f)}(\beta_1^{-1}(\log r)))))\cdot\beta_1(\exp(T_{W(g)}(2(\beta_2^{-1}(\log r)))))}=0. \tag{14}$$

If either $\beta_1(r)=C\exp(\alpha_2(r))$, where C is any positive constant or $\exp(\alpha_2(r))>\beta_1(r)$, then

$$\underline{\lim_{r\to+\infty}}\frac{\{\alpha_1(\exp(T_{f(g)}(\beta_2^{-1}(\log r))))\}^2}{\exp(\alpha_1(\exp(T_{W(f)}(\beta_1^{-1}(\log r)))))\cdot\beta_1(\exp(T_{W(g)}(2(\beta_2^{-1}(\log r)))))}=0. \tag{15}$$

Proof. In view of (3), it follows for all sufficiently large values of r that

$$\exp(\alpha_1(\exp(T_{W(f)}(\beta_1^{-1}(\log r))))) \geq r^{(\lambda_{(\alpha_1,\beta_1)}[f]-\varepsilon)}. \tag{16}$$

As $\varrho_{(\alpha_2,\beta_2)}[g] < \lambda_{(\alpha_1,\beta_1)}[f]$ we can choose $\varepsilon(>0)$ in such a way that

$$\varrho_{(\alpha_2,\beta_2)}[g] + \varepsilon < \lambda_{(\alpha_1,\beta_1)}[f] - \varepsilon. \tag{17}$$

Now from (4)we have for all sufficiently large values of r that

$$\frac{\alpha_1(\exp(T_{f(g)}(\beta_2^{-1}(\log r)))}{\beta_1(\exp(T_{W(g)}(2(\beta_2^{-1}(\log r)))))} \leq \frac{(1+o(1))(\varrho_{(\alpha_1,\beta_1)}[f]+\varepsilon)\beta_1(M_g(\beta_2^{-1}(\log r)))}{\beta_1(\exp(T_{W(g)}(2(\beta_2^{-1}(\log r)))))}.$$

Since $\beta_1 \in L_2$, in view of $\log^+ M_g(r) \leq 3T_g(2r)$ {cf. [4]}, we get from above

$$\frac{\alpha_1(\exp(T_{f(g)}(\beta_2^{-1}(\log r)))}{\beta_1(\exp(T_{W(g)}(2(\beta_2^{-1}(\log r)))))} \leq \frac{(1+o(1))(\varrho_{(\alpha_1,\beta_1)}[f]+\varepsilon)\beta_1(\exp(T_g(2(\beta_2^{-1}(\log r)))))}{\beta_1(\exp(T_{W(g)}(2(\beta_2^{-1}(\log r)))))}.$$

Since $\varepsilon > 0$ is arbitrary, so

$$\varlimsup_{r\to+\infty} \frac{\alpha_1(\exp(T_{f(g)}(\beta_2^{-1}(\log r)))}{\beta_1(\exp(T_{W(g)}(2(\beta_2^{-1}(\log r)))))} \leq \varrho_{(\alpha_1,\beta_1)}[f]. \tag{18}$$

If $\alpha_2(\beta_1^{-1}(r)) \in L_1$ then (4) implies

$$\exp(\alpha_2(\beta_1^{-1}(\alpha_1(\exp(T_{f(g)}(\beta_2^{-1}(\log r))))))) \leqslant r^{(1+o(1))(\varrho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}. \tag{19}$$

Now combining (16), (17), (18) and (19) we get

$$\varlimsup_{r\to+\infty} \frac{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(\exp(T_{f(g)}(\beta_2^{-1}(\log r)))))) \cdot \alpha_1(\exp(T_{f(g)}(\beta_2^{-1}(\log r))))}{\exp(\alpha_1(\exp(T_{W(f)}(\beta_1^{-1}(\log r))))) \cdot \beta_1(\exp(T_{W(g)}(2(\beta_2^{-1}(\log r)))))} =$$

$$= \lim_{r\to+\infty} \frac{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(\exp(T_{f(g)}(\beta_2^{-1}(\log r))))))}{\exp(\alpha_1(\exp(T_{W(f)}(\beta_1^{-1}(\log r)))))} \cdot \limsup_{r\to+\infty} \frac{\alpha_1(\exp(T_{f(g)}(\beta_2^{-1}(\log r)))}{\beta_1(\exp(T_{W(g)}(2(\beta_2^{-1}(\log r)))))} \leq$$

$$\leq \lim_{r\to+\infty} \frac{\varrho_{(\alpha_1,\beta_1)}[f] \cdot r^{(1+o(1))(\varrho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}}{r^{(\lambda_{(\alpha_1,\beta_1)}[f]-\varepsilon)}} = 0,$$

i.e., (14) is proved.

If $\beta_1(r) = C\exp(\alpha_2(r))$ then from (4) as above we have

$$\alpha_1(\exp(T_{f(g)}(\beta_2^{-1}(\log r)))) \leqslant C(1+o(1))(\varrho_{(\alpha_1,\beta_1)}[f]+\varepsilon)r^{(\varrho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}. \tag{20}$$

Therefore combining (16), (17), (18) and (20) we get

$$\varlimsup_{r\to+\infty} \frac{\{\alpha_1(\exp(T_{f(g)}(\beta_2^{-1}(\log r))))\}^2}{\exp(\alpha_1(\exp(T_{W(f)}(\beta_1^{-1}(\log r))))) \cdot \beta_1(\exp(T_{W(g)}(2(\beta_2^{-1}(\log r)))))} =$$

$$= \lim_{r\to+\infty} \frac{\alpha_1(\exp(T_{f(g)}(\beta_2^{-1}(\log r))))}{\exp(\alpha_1(\exp(T_{W(f)}(\beta_1^{-1}(\log r)))))} \cdot \varlimsup_{r\to+\infty} \frac{\alpha_1(\exp(T_{f(g)}(\beta_2^{-1}(\log r)))}{\beta_1(\exp(T_{W(g)}(2(\beta_2^{-1}(\log r)))))} \leq$$

$$\leq \lim_{r\to+\infty} \frac{\varrho_{(\alpha_1,\beta_1)}[f] \cdot C(1+o(1))(\varrho_{(\alpha_1,\beta_1)}[f]+\varepsilon)r^{(\varrho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}}{r^{(\lambda_{(\alpha_1,\beta_1)}[f]-\varepsilon)}} = 0$$

i.e., (15) is proved.

Lastly, if $\exp(\alpha_2(r)) > \beta_1(r)$ then as above we have from (4)

$$\alpha_1(\exp(T_{f(g)}(\beta_2^{-1}(\log r)))) \leqslant (1+o(1))(\varrho_{(\alpha_1,\beta_1)}[f]+\varepsilon)r^{(\varrho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}. \tag{21}$$

So combining (16), (17), (18) and (21) we get

$$\overline{\lim_{r\to+\infty}} \frac{\{\alpha_1(\exp(T_{f(g)}(\beta_2^{-1}(\log r))))\}^2}{\exp(\alpha_1(\exp(T_{W(f)}(\beta_1^{-1}(\log r))))) \cdot \beta_1(\exp(T_{W(g)}(2(\beta_2^{-1}(\log r)))))} =$$

$$= \lim_{r\to+\infty} \frac{\alpha_1(\exp(T_{f(g)}(\beta_2^{-1}(\log r))))}{\exp(\alpha_1(\exp(T_{W(f)}(\beta_1^{-1}(\log r)))))} \cdot \overline{\lim_{r\to+\infty}} \frac{\alpha_1(\exp(T_{f(g)}(\beta_2^{-1}(\log r)))}{\beta_1(\exp(T_{W(g)}(2(\beta_2^{-1}(\log r)))))}$$

$$\leq \lim_{r\to+\infty} \frac{\varrho_{(\alpha_1,\beta_1)}[f]\cdot(1+o(1))(\varrho_{(\alpha_1,\beta_1)}[f]+\varepsilon)r^{(\varrho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}}{r^{(\lambda_{(\alpha_1,\beta_1)}[f]-\varepsilon)}} = 0,$$

i.e. (15) is proved again. The proof of Theorem 4 is completed. □

Theorem 5. *Let f be meromorphic and g be a transcendental entire function having the maximum deficiency sum such that $\lambda_{(\alpha_1,\beta_1)}[f] < \infty$ ($\beta_1 \in L_2$), $\lambda_{(\alpha_2,\beta_2)}[g] > 0$ ($\alpha_2 \in L_2$) and $\varrho_{(\alpha_3,\beta_3)}[f(g)] < \infty$. Then*

$$\overline{\lim_{r\to+\infty}} \frac{\alpha_1(\exp(T_{f(g)}(\beta_2^{-1}(\log r))) \cdot \alpha_3(\exp(T_{f(g)}(\beta_3^{-1}(r))))}{\beta_1(\exp(T_{W(g)}(2(\beta_2^{-1}(\log r))))) \cdot \alpha_2(\exp(T_{W(g)}(\beta_2^{-1}(r))))} \leq \frac{\varrho_{(\alpha_3,\beta_3)}[f(g)]\cdot\varrho_{(\alpha_1,\beta_1)}[f]}{\lambda_{(\alpha_2,\beta_2)}[g]}.$$

Proof. For all sufficiently large values of r we have

$$\alpha_3(\exp(T_{f(g)}(\beta_3^{-1}(r)))) \leq (\varrho_{(\alpha_3,\beta_3)}[f(g)]+\varepsilon)r. \tag{22}$$

Again for all sufficiently large values of r it follows that

$$\alpha_2(\exp(T_{W(g)}(\beta_2^{-1}(r)))) \geq (\lambda_{(\alpha_2,\beta_2)}[g]-\varepsilon)r. \tag{23}$$

Now combining (22) and (23) we have for all sufficiently large values of r that

$$\frac{\alpha_3(\exp(T_{f(g)}(\beta_3^{-1}(r))))}{\alpha_2(\exp(T_{W(g)}(\beta_2^{-1}(r))))} \leq \frac{\varrho_{(\alpha_3,\beta_3)}[f(g)]+\varepsilon}{\lambda_{(\alpha_2,\beta_2)}[g]-\varepsilon}.$$

As $\varepsilon > 0$ is arbitrary we get from above that $\overline{\lim\limits_{r\to+\infty}} \dfrac{\alpha_3(\exp(T_{f(g)}(\beta_3^{-1}(r))))}{\alpha_2(\exp(T_{W(g)}(\beta_2^{-1}(r))))} \leq \dfrac{\varrho_{(\alpha_3,\beta_3)}[f(g)]}{\lambda_{(\alpha_2,\beta_2)}[g]}$. Hence and from (18) we obtain that

$$\overline{\lim_{r\to+\infty}} \frac{\alpha_1(\exp(T_{f(g)}(\beta_2^{-1}(\log r))) \cdot \alpha_3(\exp(T_{f(g)}(\beta_3^{-1}(r))))}{\beta_1(\exp(T_{W(g)}(2(\beta_2^{-1}(\log r))))) \cdot \alpha_2(\exp(T_{W(g)}(\beta_2^{-1}(r))))} \leq$$

$$\leq \overline{\lim_{r\to+\infty}} \frac{\alpha_1(\exp(T_{f(g)}(\beta_2^{-1}(\log r)))}{\beta_1(\exp(T_{W(g)}(2(\beta_2^{-1}(\log r)))))} \cdot \overline{\lim_{r\to+\infty}} \frac{\alpha_3(\exp(T_{f(g)}(\beta_3^{-1}(r))))}{\alpha_2(\exp(T_{W(g)}(\beta_2^{-1}(r))))} \leq$$

$$\leq \frac{\varrho_{(\alpha_3,\beta_3)}[f(g)]\cdot\varrho_{(\alpha_1,\beta_1)}[f]}{\lambda_{(\alpha_2,\beta_2)}[g]}.$$

Hence the theorem follows. □

Theorem 6. *Let f be a transcendental meromorphic function with $\sum\limits_{a\neq\infty}\delta(a;f)+\delta(\infty;f)=2$ and g be entire such that $\varrho_{(\alpha_1,\beta_1)}[f]<+\infty$ and $\lambda_{(\alpha_3,\beta_3)}[f(g)]=+\infty$. Then*

$$\lim_{r\to+\infty}\frac{\alpha_3(\exp(T_{f(g)}(r)))}{\alpha_1(\exp(T_{W(f)}(\beta_1^{-1}(\beta_3(r)))))}=\infty.$$

Proof. Let us suppose that the conclusion of the theorem do not hold. Then we can find a constant $\Delta>0$ such that for a sequence of values of r tending to infinity

$$\alpha_3(\exp(T_{f(g)}(r)))\leq\Delta\cdot\alpha_1(\exp(T_{W(f)}(\beta_1^{-1}(\beta_3(r))))). \tag{24}$$

Again from the definition of $\varrho_{(\alpha_1,\beta_1)}[W(f)]$ and in view of Lemma 4, it follows for all sufficiently large values of r that $\alpha_1(\exp(T_{W(f)}(\beta_1^{-1}(\beta_3(r)))))\leq(\varrho_{(\alpha_1,\beta_1)}[W(f)]+\varepsilon)\beta_3(r)$

$$i.e.,\ \alpha_1(\exp(T_{W(f)}(\beta_1^{-1}(\beta_3(r)))))\leq(\varrho_{(\alpha_1,\beta_1)}[f]+\varepsilon)\beta_3(r). \tag{25}$$

Thus from (24) and (25), we have for a sequence of values of $r\to+\infty$ that

$$\alpha_3(\exp(T_{f(g)}(r)))\leq\Delta(\varrho_{(\alpha_1,\beta_1)}[f]+\varepsilon)\beta_3(r),$$

i.e., $\underline{\lim}_{r\to+\infty}\dfrac{\alpha_3(\exp(T_{f(g)}(r)))}{\beta_3(r)}=\lambda_{(\alpha_3,\beta_3)}[f(g)]\leq\Delta(\varrho_{(\alpha_1,\beta_1)}[f]+\varepsilon)<+\infty$. This is a contradiction. Thus the theorem follows. □

Remark 8. If we take "$\sum\limits_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2$" and "$\varrho_{(\alpha_1,\beta_1)}[g]<+\infty$" instead of "$\sum\limits_{a\neq\infty}\delta(a;f)+\delta(\infty;f)=2$" and "$\varrho_{(\alpha_1,\beta_1)}[f]<+\infty$" and other conditions remain same, the conclusion of Theorem 6 remains true with "$\alpha_1(\exp(T_{W(f)}(\beta_1^{-1}(\beta_3(r)))))$" replaced by "$\alpha_1(\exp(T_{W(g)}(\beta_1^{-1}(\beta_3(r)))))$" in the denominator.

Remark 9. Theorem 6 and Remark 8 are also valid with "limit superior" instead of "limit" if "$\lambda_{(\alpha_3,\beta_3)}[f(g)]=+\infty$" is replaced by "$\varrho_{(\alpha_3,\beta_3)}[f(g)]=+\infty$" and the other conditions remain the same.

Using Definition 1 and Lemma 4, one can easily proof the following theorem and therefore its proof is omitted:

Theorem 7. *Let f be a transcendental meromorphic function with $\sum\limits_{a\neq\infty}\delta(a;f)+\delta(\infty;f)=2$ and g be an entire function such $0<\lambda_{(\alpha_1,\beta_1)}[f(g)]\leq\varrho_{(\alpha_1,\beta_1)}[f(g)]<+\infty$ and $0<\lambda_{(\alpha_2,\beta_2)}[f]\leq\varrho_{(\alpha_2,\beta_2)}[f]<+\infty$. Then*

$$\frac{\lambda_{(\alpha_1,\beta_1)}[f\circ g]}{\varrho_{(\alpha_2,\beta_2)}[f]}\leq\underline{\lim}_{r\to+\infty}\frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_2(\exp(T_{W(f)}(\beta_2^{-1}(\beta_1(r)))))}\leq\min\left\{\frac{\lambda_{(\alpha_1,\beta_1)}[f\circ g]}{\lambda_{(\alpha_2,\beta_2)}[f]},\frac{\varrho_{(\alpha_1,\beta_1)}[f\circ g]}{\varrho_{(\alpha_2,\beta_2)}[f]}\right\}\leq$$
$$\leq\max\left\{\frac{\lambda_{(\alpha_1,\beta_1)}[f\circ g]}{\lambda_{(\alpha_2,\beta_2)}[f]},\frac{\varrho_{(\alpha_1,\beta_1)}[f\circ g]}{\varrho_{(\alpha_2,\beta_2)}[f]}\right\}\leq\overline{\lim}_{r\to+\infty}\frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\alpha_2(\exp(T_{W(f)}(\beta_2^{-1}(\beta_1(r)))))}\leq\frac{\varrho_{(\alpha_1,\beta_1)}[f\circ g]}{\lambda_{(\alpha_2,\beta_2)}[f]}.$$

Remark 10. If we take "$\sum\limits_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2$" and "$0<\lambda_{(\alpha_2,\beta_2)}[g]\leq\varrho_{(\alpha_2,\beta_2)}[g]<+\infty$" instead of "$\sum\limits_{a\neq\infty}\delta(a;f)+\delta(\infty;f)=2$" and "$0<\lambda_{(\alpha_2,\beta_2)}[f]\leq\varrho_{(\alpha_2,\beta_2)}[f]<+\infty$" and other conditions remain same, the conclusion of Theorem 7 remains true with "$\varrho_{(\alpha_2,\beta_2)}[f]$", "$\lambda_{(\alpha_2,\beta_2)}[f]$" and "$\alpha_2(\exp(T_{W(f)}(\beta_2^{-1}(\beta_1(r)))))$" replaced by "$\varrho_{(\alpha_2,\beta_2)}[g]$", "$\lambda_{(\alpha_2,\beta_2)}[g]$" and "$\alpha_2(\exp(T_{W(g)}(\beta_2^{-1}(\beta_1(r)))))$" respectively in the denominator.

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T. Biswas: Rajbari, Rabindrapally, R. N. Tagore Road P.O. Krishnagar
Dist-Nadia, 741101, West Bengal, India
tanmaybiswas_math@rediffmail.com

C. Biswas: Department of Mathematics,Nabadwip Vidyasagar College
Nabadwip, Dist.- Nadia, 741302, West Bengal, India
chinmay.shib@gmail.com

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