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BITLYAN-GOL'DBERG TYPE INEQUALITY FOR ENTIRE FUNCTIONS AND DIAGONAL MAXIMAL TERM

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We obtain an analogue of Wiman-Bitlyan-Gol'dberg type inequality for entire $f\colon \mathbb{C}^p \to \mathbb{C}$ from the class $\mathcal{E}^p(\lambda)$ of functions represented by gap power series of the form

$$f(z)=\sum_{k=0}^{+\infty} P_k(z), \quad z\in\mathbb{C}^p.$$

Here $P_0(z)\equiv a_0\in\mathbb{C}$, $P_k(z)=\sum_{\|n\|=\lambda_k} a_n z^n$ is a homogeneous polynomial of degree $\lambda_k\in\mathbb{Z}_+$, and $0=\lambda_0<\lambda_k\uparrow+\infty$ $(1\le k\uparrow+\infty)$, $\lambda=(\lambda_k)$. We consider the exhaustion of the space $\mathbb{C}^p$ by the system $(\mathbf{G}_r)_{r\ge 0}$ of a bounded complete multiple-circular domains $\mathbf{G}_r$ with the center at the point $\mathbf{0}=(0,\dots,0)\in\mathbb{C}^p$. Define $M(r,f)=\max\{|f(z)|\colon z\in\overline{G}_r\}$, $\mu(r,f)=\max\{|P_k(z))|\colon z\in\overline{G}_r\}$. Let $\mathcal{L}$ be the class of positive continuous functions $\psi\colon\mathbb{R}_+\to\mathbb{R}_+$ such that $\int_0^{+\infty}\frac{dx}{\psi(x)}<+\infty$, $n(t)=\sum_{\lambda_k\le t}1$ the counting function of the sequence (λ_k) for $t\ge 0$. The following statement is proved: *If a sequence $\lambda=(\lambda_k)$ satisfy the condition*

$$(\exists p_1\in(0,+\infty))(\exists t_0>0)(\forall t\ge t_0)\colon\quad n(t+\sqrt{\psi(t)})-n(t-\sqrt{\psi(t)})\le t^{p_1}$$

for some function $\psi\in\mathcal{L}$, then for every entire function $f\in\mathcal{E}^p(\lambda)$, $p\ge 2$ and for any $\varepsilon>0$ there exist a constant $C=C(\varepsilon,f)>0$ and a set $E=E(\varepsilon,f)\subset[1,+\infty)$ of finite logarithmic measure such that the inequality

$$M(r,f)\le Cm(r,f)(\ln m(r,f))^{p_1}(\ln\ln m(r,f))^{p_1+\varepsilon}$$

holds for all $r\in[1,+\infty]\setminus E$.

The obtained inequality is sharp in general. In the case $\lambda_k\equiv k$, $p=2$ we have $p_1=1/2+\varepsilon$, therefore from obtained statement we get the assertion on the Bitlyan-Gol'dberg inequality (1959), and fot $p=1$ the classical Wiman inequality it follows.

1. Introduction. We use the following standard notation. Let $\mathbb{C}^p$ be the p-dimensional $(p\ge 1)$ a complex vector space, $\mathbb{Z}_+^p=(\mathbb{N}\cup\{0\})^p$, $z^n=z_1^{n_1}\cdots z_p^{n_p}$, $\|n\|=n_1+\dots+n_p$ for $n=(n_1,\dots,n_p)\in\mathbb{Z}_+^p$ and $z=(z_1,\dots,z_p)\in\mathbb{C}^p$, $\mathbb{R}_+=[0,+\infty)$. By $\mathcal{E}^p(\lambda)$ we denote of the class of entire functions $f\colon\mathbb{C}^p\to\mathbb{C}$, (i.e., entire functions of p complex variables), represented by power series of the form

$$f(z)=\sum_{k=0}^{+\infty} P_k(z), \quad z\in\mathbb{C}^p. \tag{1}$$

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Here $P_0(z) \equiv a_0 \in \mathbb{C}$, $P_k(z) = \sum_{\|n\|=\lambda_k} a_n z^n$ is a homogeneous polynomial of degree $\lambda_k \in \mathbb{Z}_+$, and $0 = \lambda_0 < \lambda_k \uparrow +\infty$ $(1 \leq k \uparrow +\infty)$, $\lambda = (\lambda_k)$. In the case $\lambda_k \equiv k$ $(k \geq 0)$ we obtain the class of all entire functions of p complex variables. Denote by $\mathcal{E}^p$; $\mathcal{E}^1$, $\mathcal{E}^1(\lambda)$ the classes of entire functions of one variable and entire functions represented by gap power series of the form

$$f(z) = a_0 + \sum_{k=1}^{+\infty} a_k z^{\lambda_k}, \quad z \in \mathbb{C}, \tag{2}$$

respectively.

According to [1] we consider the *exhaustion of the space* $\mathbb{C}^p$ by a system $(\mathbf{G}_r)_{r\geq 0}$ of bounded complete multiple-circular domains with the center at the point $\mathbf{0} = (0, \ldots, 0) \in \mathbb{C}^p$. Actually, we assume that this system satisfies the conditions:

i) $\bigcup\limits_{r\geq 0} \mathbf{G}_r = \mathbb{C}^p$;

ii) $(\forall r_1 < r_2)\colon \mathbf{G}_{r_1} \subset \mathbf{G}_{r_2}$;

iii) $(z_1, \ldots, z_p) \in \mathbf{G}_1 \Longleftrightarrow (\forall\ r > 0) : (rz_1, \ldots, rz_p) \in \mathbf{G}_r$;

iv) $(z_1, \ldots z_p) \in \mathbf{G}_r \Longrightarrow (\forall\ \theta = (\theta_1, \ldots, \theta_p) \in \mathbb{R}^p) : (z_1 e^{i\theta_1}, \ldots, z_p e^{i\theta_p}) \in \mathbf{G}_r$.

By $\mathbb{G} = \{\mathbf{G} = (\mathbf{G}_r)_{r\geq 0}\colon$ i)—iv)$\}$ we denote the class the systems of such domains. Remark that following system $\mathbf{G} = (\mathbf{G}_r)_{r\geq 0}$ of the domains $\mathbf{G}_r$ is contained in the class $\mathbb{G}$:

i) $\mathbf{G}_r = C_{r,a} = \{(z_1, \ldots, z_p) \in \mathbb{C}^p :\ |z_j| < a_j r,\ 1 \leq j \leq p\}$;

ii) $\mathbf{G}_r = \mathbb{B}_{r,a} = \{(z_1, \ldots, z_p) \in \mathbb{C}^p :\ a_1|z_1|^2 + \ldots + a_p|z_p|^2 < r^2\}$;

iii) $\mathbf{G}_r = \Pi_{r,a} = \{(z_1, \ldots, z_p) \in \mathbb{C}^p :\ a_1|z_1| + \ldots + a_p|z_p| < r\}$;

iv) $\mathbf{G}_r = \{(z_1, \ldots, z_p) \in \mathbb{C}^p : |z_1|^{a_1} \cdot \ldots \cdot |z_2|^{a_p} < r^{a_1+\cdots+a_p}\}$;

where $a = (a_1, \ldots, a_p)$, $a_j > 0$ $(1 \leq j \leq p)$, $r > 0$.

For $r > 0$ and an entire function $f \in \mathcal{E}^1(\lambda)$ we denote by $M_f(r) = \max\{|f(z)| : |z| = r\}$ the maximum modulus, and by $\mu_f(r) = \max\{|a_k|r^{\lambda_k} : k \geq 0\}$ the maximal term of power series (2). For $r > 0$ and an entire function $f \in \mathcal{E}^p(\lambda)$ of the form (1) we denote

$$M(r, f) = \max\{|f(z)| : z \in \overline{\mathbf{G}}_r\}, \quad m_k(r, f) = \max\{|P_k(z)| : z \in \overline{\mathbf{G}}_r\} \quad (k \geq 0).$$

By the maximum modulus principle there exists a point $z^{(k)} = (z_1^{(k)}, \ldots, z_p^{(k)}) \in \partial\mathbf{G}_r$ such that $m_k(r, f) = |P_k(z^{(k)})|$. The definition of $\mathbf{G}_r$ implies $s^{(k)} = (s_1^{(k)}, \ldots, s_p^{(k)}) \stackrel{def}{=} \frac{z^{(k)}}{r} \in \partial\mathbf{G}_1$, but $P_k(z)$ is a homogeneous polynomial. Hence $P_k(z^{(k)}) = r^{\lambda_k} P_k(s^{(k)})$. Thus $|P_k(s^{(k)})| = \max\{|P_k(z)|\colon z \in \overline{\mathbf{G}}_1\} = m_k(1, f)$ and therefore $s^{(k)}$ does not depend on r. So,

$$m_k(r, f) = r^{\lambda_k}|P_k(s^{(k)})| \quad (r > 0,\ k \geq 0).$$

According to [1] define now the *diagonal maximal term of the series* (1)

$$m(r, f) \stackrel{def}{=} \max\{m_k(r, f) : k \geq 0\} = \max\{r^{\lambda_k} m_k(1, f) : k \geq 0\}.$$

We remark that $m(r, f) \equiv \mu_f(r)$ in the case $p = 1$.

Let $n(t)=\sum_{\lambda_k\le t}1$ be the *counting function* of the sequence $\lambda=(\lambda_k)$. From Theorem 1 in the paper [2], proved for entire Dirichlet series, it follows such a statement. If a sequence $\lambda=(\lambda_k)$ satisfies the condition

$$\overline{\lim_{t\to+\infty}}\frac{\ln\big(n(t+\sqrt{\psi(t)})-n(t-\sqrt{\psi(t)})\big)}{\ln t}\le p_1<+\infty \tag{3}$$

for some positive function $\psi\colon\mathbb{R}_+\to\mathbb{R}_+$ such that $\int_0^{+\infty}dt/\psi(t)<+\infty$, then for every entire function $f\in\mathcal{E}^1(\lambda)$ and any $\varepsilon>0$ there exists a set $E=E(\varepsilon,f)\subset[1,+\infty)$ of finite logarithmic measure (i.e., $\text{ln-meas}(E)\overset{def}{=}\int_E d\ln r<+\infty$) such that

$$M_f(r)\le C\mu_f(r)(\ln\mu_f(r))^{p_1+\varepsilon} \tag{4}$$

holds for all $r\in[1,+\infty)\setminus E$. Here C is some constant depending only on f and ε. Hence, in particular, it follows (see also [3, 4]), that if

$$(\exists\triangle\in(0,+\infty))(\exists\varrho\in[1/2,1])(\exists D>0)(\exists t_0>0)(\forall t>t_0)\colon |n(t)-\triangle t^{\varrho}|\le D, \tag{5}$$

then inequality (4) holds with $p_1=(2\varrho-1)/2$, because in this case condition (3) is satisfied with $p_1=(2\varrho-1)/2$. In the case $f\in\mathcal{E}^1$, i.e. $\lambda_k\equiv k$ $(k\ge0)$, condition (5) holds with $\varrho=1$. Therefore, inequality (4) holds with $p_1=1/2$, i.e., we have the classical Wiman's inequality (see [5, 6, 7]).

In [2] it is also proved that for every sequence $\lambda=(\lambda_k)$ such that there exists a continuous positive increasing to $+\infty$ in the interval $[0,+\infty)$ function ψ satisfying $\psi(t)=O(t\ln t\ln^2\ln t)$ $(t\to+\infty)$, $\int\limits_0^{+\infty}\frac{dt}{\psi(t)}<+\infty$ and

$$(\exists p_1>0)\colon\qquad \underline{\lim}_{t\to+\infty} t^{-p_1}\Big(n(t+\sqrt{\psi(t)})-n(t-\sqrt{\psi(t)})\Big)>0, \tag{6}$$

there exists an entire function $f\in\mathcal{E}^1(\lambda)$ such that

$$\frac{M_f(r)}{\mu_f(r)}(\ln\mu_f(r))^{-p_1}\to+\infty\quad(r\to+\infty). \tag{7}$$

Condition (5) implies that (6) holds with $p_1=(2\varrho-1)/2$. Thus, some entire function satisfies relation (7) with $p_1=(2\varrho-1)/2$. It follows from the foregoing that if condition (5) is satisfied, then there exists a function $f\in\mathcal{E}^1(\lambda)$ such that relations (4) and (7) hold with $p_1=(2\varrho-1)/2$. In particular, for some entire function $f\in\mathcal{E}^1$ we have

$$\frac{M_f(r)}{\mu_f(r)\sqrt{\ln\mu_f(r)}}\to+\infty\quad(r\to+\infty).$$

This means that we cannot replace $(2\varrho-1)/2$ in (4) by a smaller number. Moreover, we cannot even replace $\varepsilon>0$ in (4) on $\varepsilon=0$. Hence we get in Wiman's inequality for the class of all entire functions $\mathcal{E}^1$ the number $\frac12$ cannot be replaced by a smaller number. Moreover, $\varepsilon>0$ cannot be replaced by $\varepsilon=0$. We note also that $M_f(r)\sim\sqrt{2\pi}\mu_f(r)\sqrt{\ln\mu_f(r)}$ $(r\to+\infty)$ for the entire function $f(z)=e^z$.

Theorem on Wiman's type inequality with diagonal maximal term of the series for the class $\mathcal{E}^2$ and exhausting $\mathbb{C}^2$ by an arbitrary system of complete multiple-circular domains is proved in the paper [1].

Theorem A ([1, p.33, Theorem 3]). *For every entire function $f \in \mathcal{E}^2$ and for any $\varepsilon > 0$ there exist a constant $C = C(\varepsilon, f) > 0$ and a set $E \subset [1, +\infty]$ of finite logarithmic measure such that the inequality*

$$M(r, f) \le C \cdot m(r, f)(\ln m(r, f))^{\frac{1}{2}+\varepsilon}$$

holds for all $r \in [1, +\infty] \setminus E$.

In this paper we prove analogues of cited above results ([2]) for the class $\mathcal{E}^p(\lambda)$. Obtained results in particular contain the result of Theorem A and their sharpness is also proved.

Note, that in [1] it is proved an analogue of the Wiman inequality for maximum modulus on bi-circle $M(r, f) = \max\{|f(z_1, z_2)|\colon |z_1| = r_1, |z_2| = r_2\}$ and maximal term $\mu(r, f) = \max\{|a_n|r^n : n \in \mathbb{Z}_+^2\}, r = (r_1, r_2) \in \mathbb{R}_+^2$ with 3/2 instead of 1/2 in (8). A. Schumitski ([8, 9]), P. Fenton ([10]), O. Skaskiv and O. Trakalo ([11]) and some others authors have improved Bitlyan and Goldberg's result as in the specification of inequality, and in the specification of describing exceptional set, and also established analogues Wiman's inequality for other classes analytic functions of several variables (see also [12]–[26]).

2. Main results.

2.1. Wiman's type inequality for entire gap power series and diagonal maximal term. Let $\mathcal{L}$ be the class of positive continuous functions $\psi\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $\int_0^{+\infty} \frac{dx}{\psi(x)} < +\infty$.

First we prove such assertions.

Theorem 1. *If a sequence $\lambda = (\lambda_k)$ satisfies the condition*

$$(\exists p_1 \in (0, +\infty))(\exists t_0 > 0)(\forall t \ge t_0)\colon \quad n(t + \sqrt{\psi(t)}) - n(t - \sqrt{\psi(t)}) \le t^{p_1} \tag{8}$$

for some function $\psi \in \mathcal{L}$, then for every entire function $f \in \mathcal{E}^p(\lambda)$, $p \ge 2$ and for any $\varepsilon > 0$ there exist a constant $C = C(\varepsilon, f) > 0$ and a set $E = E(\varepsilon, f) \subset [1, +\infty)$ of finite logarithmic measure such that the inequality

$$M(r, f) \le Cm(r, f)(\ln m(r, f))^{p_1}(\ln\ln m(r, f))^{p_1+\varepsilon}$$

holds for all $r \in [1, +\infty] \setminus E$.

Theorem 2. *Let $\psi \in \mathcal{L}$ be an increasing in interval $[0, +\infty)$ function such that $\psi(t) = O(t \ln t \ln^2 \ln t)$ $(t \to +\infty)$, and for a sequence $\lambda = (\lambda_k)$ the condition*

$$(\exists p_1 > 0)(\exists C_1 > 0)(\exists t_0 > 0)(\forall t \ge t_0)\colon\ n(t + \sqrt{\psi(t)}) - n(t - \sqrt{\psi(t)}] \ge C_1 t^{p_1} \tag{9}$$

holds. Then for every $\varepsilon \in (0, p_1)$ there exists an entire function $f \in \mathcal{E}^p(\lambda)$ such that

$$\frac{M(r, f)}{m(r, f)\ln^{p_1} m(r, f)\ln^{p_1-\varepsilon}\ln m(r, f)} \to +\infty \quad (r \to +\infty).$$

Remark 1. In the case of assumption (5) instead of (8) and (9) the assertions of Theorems 1 and 2 were announced in [28].

Proof of Theorem 1. Let $z = rs$, $z_1 = rs_1, \dots, z_p = rs_p$, $r > 0$, (P_k) be a homogenous polynomial, so we have

$$f(z) = \sum_{k=0}^{+\infty} r^{\lambda_k} P_k(s).$$

Hence, using $m_k(1, f) = |P_k(s^{(k)})| = \max\{|P_k(z)|\colon z \in \overline{G}_1\}$, we obtain

$$M(r, f) = \max\{|f(z)|\colon z \in \overline{G}_r\} = \max\{|f(rs)|\colon s \in \overline{G}_1\} \le$$
$$\le \sum_{k=0}^{+\infty} r^{\lambda_k} \max\{|P_k(s)| : s \in \overline{G}_1\} = \sum_{k=0}^{+\infty} r^{\lambda_k} m_k(1, f) \stackrel{def}{=} H(r).$$

Here $H(r)$ is an entire function of one variable, for which condition (8) of Theorem 1 holds.

Now we need one result from [7, Theorem 1].

Let $\mathcal{I}(\nu)$ be the class of the functions $F\colon \mathbb{R} \to \mathbb{R}_+$ defined by the integral of the form

$$F(x) = \int_{\mathbb{R}_+} a(u) e^{xu} \nu(du), \tag{10}$$

where ν is a countable additive measure on the σ-algebra $\mathcal{B}(\mathbb{R}_+)$ of Borel sets on $\mathbb{R}_+$ (Borel measure) such that $\nu(\{x\colon 0 \le x \le b\}) < +\infty$ for any $b > 0$, $a\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a positive measurable function. Denote by supp ν the support of the measure ν, i.e. the closed set $E =\colon \operatorname{supp} \nu$ such that $\nu(\mathbb{R} \setminus E) = 0$ and $\nu(\{u \in \mathbb{R}\colon |u - u_0| < r\}) > 0$ for any $u_0 \in E$ and $r > 0$. For $x \in \mathbb{R}$ and $F \in \mathcal{I}(\nu)$ we set

$$\mu_*(x) = \sup\{a(u)e^{xu}\colon u \in \operatorname{supp} \nu\}, \quad \mu^*(x) = \sup\{a(u)e^{xu}\colon u \in \mathbb{R}\}.$$

Lemma 1 ([7], Theorem 1)**.** *Let $F \in \mathcal{I}(\nu)$. If*

$$(\exists \psi \in \mathcal{L})(\exists p_1 < +\infty)(\exists t_0)(\forall t \ge t_0)\colon \quad \nu(t - \sqrt{\psi(t)}, t + \sqrt{\psi(t)}] \le t^{p_1},$$

then for every $\varepsilon > 0$ there exists a set $E_1 \subset \mathbb{R}_+$ of finite Lebesgue measure, i.e. $\operatorname{meas}(E_1) := \int_{E_1} dx < +\infty$, *such that for all $x \in [0, +\infty) \setminus E$*

$$F(x) \le C\mu_*(x) \ln^{p_1} \mu_*(x) \ln_2^{p_1+\varepsilon} \mu_*(x).$$

Here C is some constant depending only on F and ε, $\ln_2 t := \ln(\ln t)$.

Let

$$\nu(E) = \sum_{k=0}^{+\infty} \delta_{\lambda_k}(E),$$

where $\delta_\lambda(E) = 1$ for $\lambda \in E$, $\delta_\lambda(E) = 0$ for $\lambda \notin E$ for every bounded set $E \subset \mathbb{R}_+$. We put $a(u) = m_k(1, f)$ for $u = \lambda_k$, $a(u) = 0$ for $u \in \mathbb{R}_+ \setminus \{\lambda_k\}$. Then

$$H(e^x) = \int_{\mathbb{R}_+} a(u) e^{ux} \nu(du), \quad \mu_h(e^x) = \mu_*(x) \quad (x > 0).$$

Now from Lemma 1 it follows

$$M(e^x, f) \le H(e^x) \le C\mu_*(x) \ln^{p_1} \mu_*(x) \ln_2^{p_1+\varepsilon/2} \mu_*(x) =$$
$$= C\mu_H(e^x) \ln^{p_1} \mu_H(e^x) \ln_2^{p_1+\varepsilon/2} \mu_H(e^x) = Cm(e^x, f) \ln^{p_1} m(e^x, f) \ln_2^{p_1+\varepsilon/2} m(e^x, f)$$

for all $x \in \mathbb{R}_+ \setminus E_1$, $\mathrm{meas}(E_1) < +\infty$. We put $r = e^x$ and denote $E = e^{E_1}$. Then we obtain that inequality (1) holds for all $r \in [1, +\infty) \setminus E$,

$$\text{ln-meas}(E) = \int_E d\ln r = \int_{E_1} dx = \mathrm{meas}(E_1) < +\infty.$$

□

Proof of Theorem 2. Remark that for an arbitrary entire $f \in \mathcal{E}^p$ of the form

$$f(z) = \sum_{k=0}^{+\infty} a_k(z_1 \cdot \ldots \cdot z_p)^{\lambda_k}, \quad z = (z_1, \ldots, z_p) \in \mathbb{C}^p, \tag{11}$$

we have

$$m_k(r, f) = \max\{|a_k||z_1 \cdot \ldots \cdot z_p|^{\lambda_k} \colon \ (z_1, \ldots, z_p) \in \overline{G}_r\}.$$

Denote $d_k = \max\{|s_1 \cdot \ldots \cdot s_p|^{\lambda_k} \colon \ (s_1, \ldots, s_p) \in \overline{G}_1\}$ $(k \geq 1)$. Then

$$m_k(r, f) = d_k a_k r^{p\lambda_k}, \quad (k \geq 1), \tag{12}$$

and

$$m(r, f) = \max\{m_k(r, f) \colon \ k \geq 1\} = \max\{d_k a_k r^{p\lambda_k} \colon \ k \geq 1\}. \tag{13}$$

We make use of the following assertion.

Lemma 2 ([7], Theorem 2)**.** *Let $\psi \in \mathcal{L}$ be such that $\psi(t) = O(t \ln t \ln_2^2 t)$ $(t \to +\infty)$ and a measure ν such that $\ln \nu(0, t] = O(t)$ $(t \to +\infty)$ and*

$$(\exists C > 0)(\exists p_1 > 0)(\exists t_0 > 0)(\forall t \geq t_0) \colon \nu(t - \sqrt{\psi(t)}, t + \sqrt{\psi(t)}] \geq C t^{p_1}. \tag{14}$$

Then for every $\varepsilon \in (0, p_1)$ there exists $F \in \mathcal{I}(\nu)$ of form (10) such that

$$\frac{F(x)}{\mu^*(x) \ln^{p_1} \mu^*(x) \ln_2^{p_1 - \varepsilon} \mu^*(x)} \to +\infty, \ (x \to +\infty). \tag{15}$$

Let again

$$\nu(E) = \sum_{k=0}^{+\infty} \delta_{\lambda_k^{(1)}}(E), \quad \lambda_k^{(1)} := p\lambda_k.$$

Condition (9) for a sequence (λ_k) holds with a function $\psi \in \mathcal{L}$ if and only if it is holds for $(b\lambda_k)$, $b > 0$, with the function $\psi_1(u) = b^2\psi(u/b)$ instead of the function ψ, and the constant Cb^{-p_1} instead of constant C. Therefore, condition (14) holds for such ν, so Lemma 2 implies that there exists $F \in \mathcal{I}(\nu)$ of form (10) such that relation (15) is valid.

Now we put $h_k := a(\lambda_k^{(1)}) = a(p\lambda_k)$. Then

$$H(r) := \sum_{k=1}^{+\infty} h_k r^{p\lambda_k} = \int_1^{+\infty} a(u) e^{u \ln r} \nu(du) = F(\ln r), \quad \mu^*(\ln r) \geq \mu_H(r) \quad (r \geq 1).$$

Thus $h \colon \mathbb{C} \to \mathbb{C}$ is an entire function of one variable. From (15) follows

$$\frac{H(r)}{\mu^*(\ln r) \ln^{p_1} \mu^*(\ln r) \ln_2^{p_1 - \varepsilon} \mu^*(\ln r)} \to +\infty \quad (r \to +\infty). \tag{16}$$

We set $a_k = h_k/d_k$ $(k \geq 0)$ and consider a function f of the form (11). Remark, that for $r > 0$ and each $z \in \partial\overline{G}_1$

$$M(r, f) \geq |f(rz)|.$$

We put $z = (1, \ldots, 1) \in \mathbb{R}^p_+$. Without loss of generality, we may suppose that $z = (1, \ldots, 1) \in \partial\overline{G}_1$. Then, for all $r > 0$ we get

$$M(r, f) \geq |f(rz)| = \sum_{k=0}^{+\infty} h_k r^{p\lambda_k} = H(r).$$

From equalities (12), (13) it follows that

$$m(r, f) = \max\{a_k d_k r^{p\lambda_k} \colon k \geq 0\} = \max\{h_k r^{p\lambda_k} \colon k \geq 0\} = \mu_H(r) \leq \mu^*(\ln r) \quad (r \geq 1).$$

Therefore, relation (16) implies

$$\frac{M(r, f)}{m(r, f) \ln^{p_1} m(r, f) \ln_2^{p_1-\varepsilon} m(r, f)} \geq \frac{H(r)}{\mu^*(\ln r) \ln^{p_1} \mu^*(\ln r) \ln_2^{p_1-\varepsilon} \mu^*(\ln r)} \to +\infty \qquad (17)$$

as $r \to +\infty$. We suppose now that $z = (1, \ldots, 1) \notin \partial\overline{G}_1$, and $r_1 > 0$ such that $z = (r_1, \ldots, r_1) \in \partial\overline{G}_1$. Consider the function $f_1(z) = f(zr_1)$ and the exhaustion by the domains $\overline{G}^0_r := \overline{G}_{rr_1}$. Then, $z = (1, \ldots, 1) \in \partial\overline{G}^0_1$,

$$M(r, f_1, G^0) = M(r, f, G), \quad m(r, f_1, G^0) = m(r, f, G).$$

Since $M(r, f) = M(r, f, G)$ and $M(r, f_1) = M(r, f_1, G^0)$ for $r > 0$,

$$\frac{M(r, f)}{m(r, f) \ln^{p_1} m(r, f) \ln_2^{p_1-\varepsilon} m(r, f)} = \frac{M(r, f_1)}{m(r, f_1) \ln^{p_1} m(r, f_1) \ln_2^{p_1-\varepsilon} m(r, f_1)} \to +\infty$$

as $r \to +\infty$, because for the function f_1 and the domains G^0_r relation (17) holds. □

2.2. Bitlyan-Goldberg type inequality and new description of exceptional set. We will adhere to the general scheme of reasoning from the previous subsection 2.1. First we get one statement containing a new estimate of the exceptional set of functions from class $\mathcal{I}(\nu)$.

Theorem 3. *Let $F \in \mathcal{I}(\nu)$ and ν be a Borel measure such that $(\exists t_0 \geq 0)(\exists c_2, c_3 > 0)(\forall T \geq t_0)(\forall t \in [t_0, T])$:*

$$\nu(T - t, T + t] \leq c_2 t + c_3. \qquad (18)$$

If h is a positive function such that $\int_0^{+\infty} h(x)dx = +\infty$ and $\ln_2^+ h(x) = o(\ln_2 F(x))$ $(x \to +\infty)$, then for each $\varepsilon > 0$ there exists a set $E_3(\varepsilon, F, h) \equiv E_3$ such that h-meas $E_3 := \int_{E_3} h(x)dx < +\infty$ and the inequality

$$F(x) \leq h(x)\mu_*(x)(\ln \mu_*(x))^{1/2+\varepsilon}$$

holds for every $x \in [0; +\infty) \setminus E_3$.

We need the following assertion.

Lemma 3 ([29]). *Let $g_1(x)$ be a positive differentiable non-decreasing on $[0;+\infty)$ function, $\psi(x)$ be a positive continuous increasing on $[0;+\infty)$ function such that $\int_0^{+\infty}\frac{dx}{\psi(x)}<+\infty$, and $h_0(x)$ be a positive local integrable on $[0;+\infty)$ function such that $\int_0^{+\infty}h_0(x)\,dx=+\infty$. Then there exists a set $E_0\subset[0;+\infty)$ such that h-meas $E_0:=\int_{E_0}h_0(x)\,dx<+\infty$ and for every $x\in[0;+\infty)\setminus E_0$*

$$g_1'(x)\le h_0(x)\psi(g_1(x)).$$

Proof of Theorem 3. We put $g(x)=\ln F(x)$. As in [29] for fixed $x\in\mathbb{R}$ we have

$$F(x)\le 2\int_{|u-g'(x)|<\sqrt{2g''(x)}}f(u)e^{xu}\nu(du).$$

Thus by the hypotheses of Theorem 3

$$F(x)\le 2\mu_*(x)\nu(g'(x)-\sqrt{2g''(x)},g'(x)+\sqrt{2g''(x)}]\le 2\mu_*(x)\Big(c_2\sqrt{2g''(x)}+c_3\Big).$$

Let for $\varepsilon_1>0$ and $\varepsilon_2>0$

$$E_1=\{x>0: g''(x)>h(x)g'(x)(\ln g'(x))^{1+\varepsilon_1},\ g'(x)\ge 2,$$
$$E_2=\{x>0: g'(x)>h(x)(g(x))^{1+\varepsilon_2},\ g(x)\ge 1\}.$$

Now we use the statement of the Lemma 3 twice. First, choosing $g_1(x)=g'(x),\ \psi(x)=2c_2^2x(\ln x)^{1+\varepsilon_1}/(2c_2^2)$, and then $g_1(x)=g(x),\ \psi(x)=x^{1+\varepsilon_2}$, we obtain that

$$h\text{-meas } E_1<+\infty,\ h\text{-meas } E_2<+\infty,$$

i.e. h-meas $E_1\cup E_2<+\infty$.

Therefore for $x\notin E:=E_1\cup E_2$ we have

$$F(x)\le 2\mu_*(x)\bigg(h(x)\big(g(x)\big)^{(1+\varepsilon_2)/2}\Big(\ln\Big(h(x)\big(g(x)\big)^{1+\varepsilon_2}\Big)\Big)^{(1+\varepsilon_1)/2}+c_3\bigg).$$

By the assumptions of Theorem 3 $\ln h(x)<(\ln F(x))^\delta$ $(x\ge x_0(\delta))$ for every $\delta>0$. Therefore,

$$\Big(\ln\Big(h(x)(g(x))^{1+\varepsilon_2}\Big)\Big)^{(1+\varepsilon_1)/2}\le$$
$$\le\big((\ln F(x))^\delta+(1+\varepsilon_2)\ln_2 F(x)\big)^{(1+\varepsilon_1)/2}\le\big(1+o(1)\big)\big(\ln F(x)\big)^{\delta(1+\varepsilon_1)/2}$$

as $x\to+\infty$. Hence

$$F(x)\le 2\mu_*(x)\bigg(h(x)\big(\ln F(x)\big)^{(1+\varepsilon)/2}+c_3\bigg)\quad (x\to+\infty,\ x\notin E),$$

where $\varepsilon=\varepsilon_2+\delta(1+\varepsilon_1)$.

We note that $h(x)\le\exp\{\ln^{1/3}F(x)\}$ $(x\to+\infty)$. Thus

$$h(x)\big(\ln F(x)\big)^{(1+\varepsilon)/2}+c_3\le\exp\{\ln^{2/3}F(x)\}\quad (x\to+\infty)$$

and as $x\to+\infty$ $(x\notin E)$

$$\ln F(x)\le\ln 2+\ln\mu_*(x)+\ln\bigg(h(x)\big(\ln F(x)\big)^{(1+\varepsilon)/2}+c_3\bigg)\le$$
$$\le\ln 2+\ln\mu_*(x)+\ln^{2/3}F(x).$$

hence $\ln F(x) \le (1+o(1))\ln \mu_*(x)$. Consequently,

$$F(x) \le 2\mu_*(x)\Big(h(x)\big(\ln F(x)\big)^{(1+\varepsilon)/2} + c_3\Big) \le 4\mu_*(x)h(x)\big(\ln \mu_*(x)\big)^{(1+\varepsilon)/2}$$

as $x \to +\infty$ $(x \notin E)$. □

We now consider the general case of entire functions from the class $\mathcal{E}^p := \mathcal{E}^p(\lambda)$ with $\lambda_k \equiv k \in \mathbb{Z}_+$.

Theorem 3 implies the following corollary.

Theorem 4. *Let $f \in \mathcal{E}^p$. If a positive local integrable on $[1;+\infty)$ function h_0 is such that $\int_1^{+\infty} h_0(r)d\ln r = +\infty$ and $\ln^+ \ln h_0(r) = o(\ln\ln m(r,f))$ $(r \to +\infty)$, then for each $\varepsilon > 0$ there exists a set $E_4(\varepsilon, f, h) \equiv E_4$ such that h_0-log-meas $E_4 := \int_{E_4} h_0(r)d\ln r < +\infty$ and*

$$M(r,f) \le h_0(r)m(r,f)(\ln m(r,f))^{1/2+\varepsilon}$$

holds for all $r \in [1,+\infty) \setminus E_4$.

Proof. First, we reason as in the proof of Theorem 1 up to the application of Lemma 1. Next, applying instead of Lemma 1 the statement of Lemma 3, we obtain the required inequality and the required description of the exceptional set, because

$$\int_{E_4} h_0(r)d\ln r = \int_{E_3} h_0(e^x)dx = \int_{E_3} h(x)dx < +\infty,$$

where the set E_4 is the image of the set E_3 by mapping $r = e^x\colon E_3 \to E_4$.

Note now, that for the sequence $\lambda_k \equiv k$ one has $n(T+t) - n(T-t) \le 2t+1$, i.e. conditions (18) take place with $c_2 = 2, c_3 = 1$. It remains to obtain from the condition $\ln^+ \ln h_0(r) = o(\ln\ln m(r,f))$ $(r \to +\infty)$ condition $\ln_2^+ h(x) = o(\ln_2 F(x))$ $(x \to +\infty)$ with $h(x) = h_0(e^x)$, $F(x) = H(e^x)$. But, by Cauchy's inequality $m(r,f) = \mu_H(r) \le H(r)$. □

If in Theorem 4 we choose $h_0(r) = \ln^\varepsilon m(r,f)$, then we immediately obtain the following statement.

Corollary. *If $f \in \mathcal{E}^p$, then for each $\varepsilon > 0$ there exists a set $E_5(\varepsilon, f, h) \equiv E_5$ such that $\int_{E_5} \ln^\varepsilon m(r,f)d\ln r < +\infty$ and the inequality*

$$M(r,f) \le m(r,f)(\ln m(r,f))^{1/2+\varepsilon}$$

holds for all $r \in [1,+\infty) \setminus E_5$.

Remark, that in the case of entire functions of one complex variables statements similarly to Theorem 4 and Corollary 1 we find in [30] (see also [31]).

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