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PROPERTIES OF SINGLE LAYER POTENTIALS FOR A PSEUDO-DIFFERENTIAL EQUATION RELATED TO A LINEAR TRANSFORMATION OF A ROTATIONALLY INVARIANT STABLE STOCHASTIC PROCESS

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We aim at determining existence conditions of single layer potentials for pseudo-differential equations related to some linear transformations of a rotationally invariant stable stochastic process in a multidimensional Euclidean space and investigating their properties as well. The carrier surface of the potential is smooth enough. We consider two main cases: the first, when this surface is bounded and closed; the second, when it is unbounded, but could be presented by an explicit equation in some coordinate system. The density of this potential is a continuous function. It is bounded with respect to the spatial variable and, probably, has an integrable singularity with respect to the time variable at zero. Classic properties of this potential, including a jump theorem of the action result of some operator (an analog of the co-normal differential) at its surface points, considered.

A rotationally invariant α-stable stochastic process in $\mathbb{R}^d$ is a Lévy process with the characteristic function of its value in the moment of time $t > 0$ defined by the expression $\exp\{-tc|\xi|^\alpha\}$, $\xi \in \mathbb{R}^d$, where $\alpha \in (0, 2]$, $c > 0$ are some constants. If $\alpha = 2$ and $c = 1/2$, we get Brownian motion and classic theory of potential. There are many different results in this case. The situation of $\alpha \in (1, 2)$ is considered in this paper. We study constant and invertible linear transformations of the rotationally invariant α-stable stochastic process. The related pseudo-differential equation is the parabolic equation of the order α of the "heat" type in which the operator with respect to the spatial variable is the process generator. The single layer potential is constructed in the same way as the single layer potential for the heat equation in the classical theory of potentials. That is, we use the fundamental solution of the equation, which is the transition probability density of the related process. In our theory, the role of the gradient operator is performed by some vector pseudo-differential operator of the order $\alpha - 1$. We have already studied the following main properties of the single layer potentials: the single layer potential is a solution of the relating equation outside of the carrier surface and the jump theorem is held. These properties can be useful to solving initial boundary value problems for the considered equations.

Introduction. A rotationally invariant α-stable ($\alpha \in (0; 2]$) stochastic process in the multidimensional Euclidean space $\mathbb{R}^d$, $d \ge 2$ is a standard Markov process $(x_0(t))_{t\ge 0}$ defined by its transition probability density

$$g_0(t, x, y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i(\xi, x-y) - t|\xi|^\alpha} d\xi, \quad t > 0, x \in \mathbb{R}^d, y \in \mathbb{R}^d.$$

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Let P be some invertible real $d \times d$-matrix. Consider the Markov process $x(t) = Px_0(t)$, $t \geq 0$. It has got the transition probability density given by the following equality

$$g(t, x, y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i(\xi, x-y) - t(Q\xi, \xi)^{\frac{\alpha}{2}}} d\xi, \tag{1}$$

where $Q = PP^T$.

Function (1) is the fundamental solution to the following pseudo-differential equation

$$\frac{\partial u(t, x)}{\partial t} = \mathbf{A}u(t, \cdot)(x), \quad t > 0, x \in \mathbb{R}^d, \tag{2}$$

where $\mathbf{A}$ is the pseudo-differential operator defined by its symbol $(-(Q\xi, \xi)^{\frac{\alpha}{2}})_{\xi \in \mathbb{R}^d}$.

In this paper, we consider the case of $\alpha \in (1; 2)$. If $\alpha = 2$, the process $(x_0(t))_{t \geq 0}$ is the standard Brownian motion and relation (2) is the well-known heat equation. In the case of $\alpha \leq 1$, we have the first order equation, what is a completely different situation.

Let S be some two-sided surface in $\mathbb{R}^d$ and $\psi(t, x)$ be some function determined on the domain $(0; +\infty) \times S$. We call the function $(v(t, x))_{t \geq 0, x \in \mathbb{R}^d}$, defined by the equality

$$v(t, x) = \int_0^t d\tau \int_S g(t - \tau, x, y)\psi(\tau, y) d\sigma_y, \tag{3}$$

the single layer potential with the density ψ on the surface S for the pseudo-differential equation (2).

The aim of this article is determining of existence conditions of the single layer potentials and researching their properties. We consider the surface S to be quite smooth, at least it belong to the class $H^{1+\gamma}$ with some $\gamma \in (0, 1)$ (see below). In this paper, two main situations are considered: the first, when the surface S is bounded and closed; the second, when it is unbounded, but for every two its points $x \in S$, $y \in S$ it holds that $\cos(\widehat{n_x, n_y}) \geq \rho_0 > 0$, where $n_x \in \mathbb{R}^d$ is a normal vector to one side of the surface S at the point $x \in S$.

The notation of the simple layer potential for the pseudo-differential equation of the type (2) (in the case $Q = \text{const} \cdot I$ with the unit matrix I) was introduced in [1], where its main properties were proved in that case. The usage of the single layer potentials for solving initial boundary value problems was considered in [1, 5, 6]. In paper [7], we obtained some results which were approximately close to ours, but we managed to consider some different analog of the gradient operator.

1. Some auxiliary results.

1.1. The function g. The function g given by formula (1) above is continuous in the domain $t > 0$, $x \in \mathbb{R}^d$, $y \in \mathbb{R}^d$ and it is uniformly continuous on each set like $t \in [\tau, +\infty)$, $x \in \mathbb{R}^d$, $y \in \mathbb{R}^d$ for $\tau > 0$.

The following estimations of the function g and its derivatives (see [3, Ch.4]):

$$|\mathbf{D}^k g(t, \cdot, y)(x)| \leq N_k \frac{t}{(t^{1/\alpha} + |y - x|)^{d+\alpha+k}}, \quad t > 0, x \in \mathbb{R}^d, y \in \mathbb{R}^d. \tag{4}$$

$$|\mathbf{D}^\varkappa g(t, \cdot, y)(x)| \leq \tilde{N}_\varkappa \frac{1}{(t^{1/\alpha} + |y - x|)^{d+\varkappa}}, \quad t > 0, x \in \mathbb{R}^d, y \in \mathbb{R}^d. \tag{5}$$

Here $\mathbf{D}^k$ denotes the differential operator of the degree $k = 0, 1, 2...$, $\mathbf{D}^\varkappa$ denotes the pseudo-differential operator defined by a homogeneous symbol $(Q_\varkappa(\xi))_{\xi\in\mathbb{R}^d}$ of a degree $\varkappa$, which have all derivatives of degrees $1 \le l \le M$ at points $\xi \ne 0$, where $M \ge 2d + \varkappa + \alpha + 1$, and satisfies the inequality $|\mathbf{D}^l Q_\varkappa(\xi)| \le C_M|\xi|^{\varkappa-l}$ for all $\xi \ne 0$, where $C_M \ge 0$ is some constant; N_k and $\tilde{N}_\varkappa$ are some positive constants.

1.2. The operator A. The action of the operator $\mathbf{A}$ defined in the introduction at smooth enough (at least which have Lipschitz continuous gradient) and bounded with its derivatives functions $\varphi(x)_{x\in\mathbb{R}^d}$ is defined by the following expression

$$\mathbf{A}\varphi(x) = \frac{q_\alpha}{(\det Q)^{1/2}} \int_{\mathbb{R}^d} (\varphi(x+z) - \varphi(x) - (\nabla\varphi(x), z))(Q^{-1}z, z)^{-\frac{d+\alpha}{2}} dz, \tag{6}$$

where $q_\alpha = \frac{\alpha\Gamma((3-\alpha)/2)\Gamma((d+\alpha)/2)}{\pi^{(d+1)/2}\Gamma(2-\alpha)}$. The value of the constant q_α can be obtained by using the operator $\mathbf{A}$ to the function $\varphi_\xi(x) = e^{i(\xi,x)}$ with some fixed $\xi \in \mathbb{R}^d$.

1.3. The operator B is an analog of the gradient. Let us define the operator $\mathbf{B}$ by its symbol $(i|\xi|^{\alpha-2}\xi)_{\xi\in\mathbb{R}^d}$. The action of the operator $\mathbf{B}$ on a bounded Lipschitz continuous function $(\varphi(x))_{x\in\mathbb{R}^d}$ is defined by the following formula:

$$\mathbf{B}\varphi(x) = \frac{q_\alpha}{\alpha} \int_{\mathbb{R}^d} \frac{(\varphi(x+z) - \varphi(x))}{|z|^{d+\alpha}} z dz,$$

where q_α is the same constant as above.

It is not difficult to obtain (see, for example, [7]) that

$$\mathbf{B}g(t, \cdot, y)(x) = \frac{1}{\alpha t}(y - x)g(t, x, y) + f(t, x, y), \quad t > 0, x \in \mathbb{R}^d, y \in \mathbb{R}^d, \tag{7}$$

where $f(t, x, y) = \frac{i}{(2\pi)^d} \int_{\mathbb{R}^d} (|\xi|^{\alpha-2}\xi - (Q\xi, \xi)^{\frac{\alpha}{2}-1}Q\xi)e^{i(\xi,x-y)-t(Q\xi,\xi)^{\frac{\alpha}{2}}} d\xi$.

Take into consideration that the function f is the action result on the function ($t > 0$, $y \in \mathbb{R}^d$ are fixed) $(g(t, x, y))_{x\in\mathbb{R}^d}$ of the pseudo-differential operator which is defined by the symbol $(i|\xi|^{\alpha-2}\xi - (Q\xi, \xi)^{\frac{\alpha}{2}-1}Q\xi)_{\xi\in\mathbb{R}^d}$. The results of the monograph [3, Ch.4] (see also (5)) lead us to the following estimation

$$|f(t, x, y)| \le \frac{C}{(t^{\frac{1}{\alpha}} + |x - y|)^{d+\alpha-1}},$$

where C is some positive constant.

Here and below, we denote by the letter C any positive constant which value does not matter. Sometimes we equip this letter with an index to indicate which parameter this constant depends on.

Let ν be some fixed vector in $\mathbb{R}^d$. We use the notation $\mathbf{B}_\nu$ for the operator $(\nu, \mathbf{B})$ which is the pseudo-differential operator with the symbol $(i|\xi|^{\alpha-2}(\xi, \nu))_{\xi\in\mathbb{R}^d}$.

1.4. The class $H^{1+\gamma}$ of surfaces. Let some surface S be given in $\mathbb{R}^d$. Suppose that there is such a constant $r_0 > 0$, that for any point $x \in S$ the part of this surface $S_{r_0}(x) = S \cap B_{r_0}(x)$ (here and below the notation $B_\delta(z)$ means the closed ball in $\mathbb{R}^d$ which has the radius $\delta > 0$ and the center placed in the point $z \in \mathbb{R}^d$) can be defined by an equation $y^d = F_x(y^1, y^2, \ldots, y^{d-1})$

in some local coordinate system with the origin in the point x. Here F_x is some single-valued function. Remember (see, for example, [2, Ch.IV, §4]) that S is called a surface of the class $H^{1+\gamma}$ with some $\gamma \in (0,1)$ if for every $x \in S$ the corresponded function F_x has continuous partial derivatives $\frac{\partial F_x}{\partial y^k}$, $k = 1, 2, \dots, d-1$ in the domain

$$\sum_{k=1}^{d-1} (y^k)^2 \le r_0^2/4$$

and they are Hölder continuous with power γ and the constant which does not depend on x.

It is clear that there exist the tangent hyperplane and the normal at each point of such kind of surfaces.

2. The single layer potential.

2.1. Existence conditions and properties. Let the surface S be smooth enough, i.e. it belongs to the class $H^{1+\gamma}$ with some fixed $\gamma \in (0;1)$, and it separates the set $\mathbb{R}^d$ into two open subsets: D_- and D_+ which satisfy the equality $\mathbb{R}^d = D_- \cup S \cup D_+$. Above we call them by "interior" and "exterior" set, respectively.

Consider the following two situations, where the surface S has some additional properties:

(A) S is bounded and closed;

(B) S is unbounded and for any two its points $x \in S$, $y \in S$ the exterior normal vectors n_x and n_y form an angle φ_{xy} such that $\cos \varphi_{xy} \ge \rho_0 > 0$.

Note that a hyperplane satisfies the property **(B)**.

Let us consider some continuous function $(\psi(t,x))_{t\ge 0, x\in S}$ so that the following inequality $|\psi(t,x)| \le K_T t^{-\beta}$ holds for all $0 < t \le T$, $x \in S$ and for each fixed $T > 0$. There $\beta < 1$ is some constant and $K_T > 0$ is constant that, possibly, depends on T.

Next we need the following useful statement.

Lemma 1. *Let the surface S satisfy the above formulated conditions. Then for every $\theta > -1$ there exists a constant $C > 0$ such that for all $t > 0$, $x \in \mathbb{R}^d$ the following inequality*

$$\int_S \frac{d\sigma_y}{(t^{1/\alpha} + |y-x|)^{d+\theta}} \le C t^{-\frac{\theta+1}{\alpha}}$$

holds.

Proof. Let S satisfy condition (A). The statement of the lemma is proved in [1]. Otherwise, if S satisfies the condition (B), let project the surface S on the tangent hyperplane to S in the point $\tilde{x}$, which is the orthogonal projection of x on S.

Then we obtain inequalities ($\tilde{y}$ is the projection of the point $y \in S$ on the mentioned hyperplane)

$$\int_S \frac{d\sigma_y}{(t^{1/\alpha} + |y-x|)^{d+\theta}} \le \int_{\mathbb{R}^{d-1}} \frac{\cos(\widehat{n_{\tilde{x}}, n_{\tilde{y}}})^{-1} d\tilde{y}}{(t^{1/\alpha} + |\tilde{y}-\tilde{x}|)^{d+\theta}} \le \frac{1}{\rho_0} \int_{\mathbb{R}^{d-1}} \frac{dz}{(t^{1/\alpha} + |z|)^{d+\theta}}.$$

Farther, the calculation of the last integral in the spherical coordinates leads us to the statement of lemma. □

In the classical case (when $\alpha = 2$) the single layer potential is a continuous function and it satisfies the corresponding parabolic differential equation in the domain $(0; +\infty) \times (\mathbb{R}^d \setminus S)$ (see [2, Ch.V]). Let us prove the analogous statement in our case.

Theorem 1. *Let a surface S in $\mathbb{R}^d$ belong to the class $H^{1+\gamma}$ with some fixed $\gamma \in (0; 1)$ and satisfy one of the conditions (A) or (B). Let a continuous function $(\psi(t, x))_{t\geq 0, x\in S}$ satisfy the inequality $|\psi(t, x)| \leq K_T t^{-\beta}$ on each set of the form $(0; T] \times S$ with some constants $\beta < 1$ and $K_T > 0$ (the last may depend on $T > 0$).*

Therefore the single layer potential (3) is correctly defined; it is the continuous function and satisfies the equation

$$\frac{\partial v(t, x)}{\partial t} = \mathbf{A}v(t, \cdot)(x)$$

for all $(t; x)$ from the domain $(0; \infty) \times (\mathbb{R}^d \setminus S)$.

Proof. Inequality (4) and the statement of Lemma 1 lead us to the inequalities

$$|v(t, x)| \leq N_0 K_T \int_0^t (t - \tau)\tau^{-\beta}\, d\tau \int_S \frac{d\sigma_y}{((t - \tau)^{1/\alpha} + |x - y|)^{d+\alpha}} \leq$$
$$\leq C_T B\left(1 - \frac{1}{\alpha}, 1 - \beta\right) t^{1-\frac{1}{\alpha}-\beta}$$

valid for all $t \in (0; T]$, $x \in \mathbb{R}^d$ and each $T > 0$. This yields to the fact that the function $(v(t, x))_{t>0, x\in\mathbb{R}^d}$ is is correctly defined and continuous.

Now, let the point $x \in \mathbb{R}^d \setminus S$ be selected and fixed. It is obvious that

$$\frac{\partial}{\partial t} v(t, x) = \int_0^t d\tau \int_S \frac{\partial}{\partial t} g(t - \tau, x, y)\psi(\tau, y) d\sigma_y + \lim_{\varepsilon\to 0+} \int_S g(\varepsilon, x, y)\psi(\tau, y) d\sigma_y.$$

In the case (A), we have the equality $\lim_{\varepsilon\to 0+} \int_S g(\varepsilon, x, y)\psi(t, y) d\sigma_y = 0$, where $t > 0$, $x \in \mathbb{R}^d \setminus S$. This relation follows from estimate (4) for $k = 0$ and the next inequality

$$\left|\int_S g(\varepsilon, x, y)\psi(t, y) d\sigma_y\right| \leq N_0 \varepsilon \frac{|S|}{(\rho(x, S))^{d+\alpha}} K_T t^{-\beta},$$

where $|S|$ is area of the surface S, $\rho(x, S)$ is the distance from the point x to the surface S.

If the condition (B) is fulfilled for the surface S we take into account the fact that the following property (see [2, Ch.IV, §4]) $0 < \varkappa_1 \leq \frac{|y-x|}{|\tilde{y}-x|} \leq \varkappa_2$, is true, where $\varkappa_i$ are some positive constants, $y \in S$, $\tilde{y}$ is the projection of the point y on the tangent hyperplane to S at the point $\tilde{x}$ ($\tilde{x}$ is the projection of x on S). Then we get the following chain of relations:

$$\left|\int_S g(\varepsilon, x, y)\psi(t, y) d\sigma_y\right| \leq N_0 K_T t^{-\beta} \varepsilon \int_S \frac{d\sigma_y}{(\varepsilon^{1/\alpha} + |x - y|)^{d+\alpha}} \leq$$
$$\leq C_T t^{-\beta} \varepsilon \int_{\mathbb{R}^{d-1}} \frac{(\cos(\widehat{n_{\tilde{x}}, n_{\tilde{y}}}))^{-1} d\tilde{y}}{(\varepsilon^{1/\alpha} + \varkappa_2\sqrt{\tilde{y}^2 + \rho(x, S)^2})^{d+\alpha}} \leq$$
$$\leq C_T t^{-\beta} \varepsilon \rho_0^{-1} \int_{\mathbb{R}^{d-1}} \frac{dz}{(z^2 + \rho(x, S)^2)^{\frac{d+\alpha}{2}}} \varkappa_2^{-(d+\alpha)} \to 0, \quad \varepsilon \to 0+.$$

Thus, the equality $\frac{\partial}{\partial t}v(t,x) = \int_0^t d\tau \int_S \frac{\partial}{\partial t}g(t-\tau,x,y)\psi(\tau,y)d\sigma_y$ holds in each case.

Now, we prove an admissibility of changing the order of integration in the integral

$$I = \int_0^t d\tau \int_S \psi(\tau,y)d\sigma_y \times$$
$$\times \int_{\mathbb{R}^d} \left[g(t-\tau,x+u,y) - g(t-\tau,x,y) - (\nabla_x g(t-\tau,x,y),u)\right](Q^{-1}u,u)^{-\frac{d+\alpha}{2}}du,$$

using representation (6) of the operator $\mathbf{A}$. Divide this integral by the sum of two integrals I_1 and I_2 of the same integrand: the inner integral in the first of them by the ball $B_\delta(0)$, and in the second by the set $\mathbb{R}^d \setminus B_\delta(0)$ (remember that $B_\delta(0) = u \in \mathbb{R}^d : |u| \le \delta)$

For small enough $\delta > 0$ and all $u \in B_\delta(0)$, $x \in \mathbb{R}^d \setminus S$, $y \in S$, $t > 0$ the following inequality

$$g(t,x+u,y) - g(t,x,y) - (\nabla_x g(t,x,y),u) \le \frac{1}{2}\sum_{i,j=1}^d \left|\frac{\partial^2 g(t,z,y)}{\partial z_i \partial z_j}\right|_{z=x+\theta(t,y)u} |u|^2,$$

holds, where $\theta = \theta(t;y) \in (0;1)$, then the absolute value of the integrand of the integral I_1 is estimated above by the expression

$$C_T \tau^{-\beta}\frac{t-\tau}{((t-\tau)^{1/\alpha}+|y-x|-\delta)^{d+\alpha+2}}|u|^{-d-\alpha+2}.$$

This expression is integrable with respect to (τ,y,u) by $(0,t)\times S\times B_\delta(0)$ for small enough $\delta > 0$. So, the integral I_1 is absolutely convergent.

The absolute value of the integrand of I_2 is estimated above by the expression

$$C_T\tau^{-\beta}\left(\frac{t-\tau}{((t-\tau)^{1/\alpha}+|x+u-y|)^{d+\alpha}} + \frac{t-\tau}{((t-\tau)^{1/\alpha}+|x-y|)^{d+\alpha}} + \right.$$
$$\left. + \frac{(t-\tau)|u|}{((t-\tau)^{1/\alpha}+|x-y|)^{d+\alpha+1}}\right)|u|^{-d-\alpha}.$$

Taking into account the estimate from Lemma 1 and the fact that $|x-y| \ge \rho(x,S) > 0$, we obtain the integrability of the second and third terms in this expression with respect to (τ,y,u) by $(0,t)\times S\times(\mathbb{R}^d\setminus B_\delta(0))$. Consider the integral of the first term and change the variable u by using the equality $x+u-y = (t-\tau)^{1/\alpha}v$. We have

$$\int_0^t \tau^{-\beta}d\tau \int_S d\sigma_y \int_{D(\tau,y)} \frac{|y-x+(t-\tau)^{1/\alpha}v|^{-d-\alpha}dv}{(1+|v|)^{d+\alpha}},$$

where $D(\tau,y) = \{v \in \mathbb{R}^d : |y-x-(t-\tau)^{1/\alpha}v| > \delta\}$. Obviously, this integral is convergent. Thus, in the integral I it is possible to arbitrarily change the order of integration. That is, for all $t > 0$, $x \in \mathbb{R}^d \setminus S$ we have the relation

$$\mathbf{A}v(t,\cdot)(x) = \int_0^t d\tau \int_S \mathbf{A}g(t-\tau,\cdot,y)(x)\psi(\tau,y)\,d\sigma_y.$$

Since the function $(g(t,x,y))_{t>0,\ x\in\mathbb{R}^d}$ for each fixed $y \in \mathbb{R}^d$ satisfies the equality $\frac{\partial}{\partial t}g(t,x,y) = \mathbf{A}g(t,\cdot,y)(x)$ throughout its domain, the statement is proved. $\square$

2.2. The jump theorem. The jump theorem takes the central place in the classical theory (for $\alpha = 2$) of the single layer potential. This theorem defines the jump of the conormal derivative of the single layer potential at the points of its carrier surface. This section is dedicated to the analogous theorem in our case ($1 < \alpha < 2$).

Lemma 2. *Let a surface S and a function $(\psi(t,x))_{t\geq 0, x\in S}$ satisfy the conditions of Theorem 1. Then for every $t > 0$, $x \in S$ the following integral ($\hat{\nu}(x) = Q^{-1}n_x$)*

$$\mathbf{B}^{dv}_{\hat{\nu}(x)} v(t,\cdot)(x) := \int_0^t d\tau \int_S \mathbf{B}_{\hat{\nu}(x)} g(t-\tau,\cdot,y)(x)\psi(\tau,y)d\sigma_y \tag{8}$$

is finite.

Proof. Use the next representation (see formula (7))

$$\mathbf{B}g(t,\cdot,y)(x) = \frac{1}{\alpha t}(y-x)g(t,x,y) + \frac{i}{(2\pi)^d}\int_{\mathbb{R}^d}(|\xi|^{\alpha-2}\xi - (Q\xi,\xi)^{\frac{\alpha}{2}-1}Q\xi)e^{i(\xi,x-y)-t(Q\xi,\xi)^{\frac{\alpha}{2}}}d\xi.$$

Divide the integral from the right-hand side of (8) by the sum of the following integrals

$$I_1 = \frac{1}{\alpha}\int_0^t \frac{d\tau}{t-\tau}\int_S (y-x,\hat{\nu}(x))g(t-\tau,x,y)\psi(\tau,y)d\sigma_y,$$

$$I_2 = \frac{i}{(2\pi)^d}\int_0^t d\tau \int_S \psi(\tau,y)d\sigma_y \int_{\mathbb{R}^d}(|\xi|^{\alpha-2}\xi - (Q\xi,\xi)^{\frac{\alpha}{2}-1}Q\xi,\hat{\nu}(x))e^{i(\xi,x-y)-t(Q\xi,\xi)^{\frac{\alpha}{2}}}d\xi.$$

We use estimate (4) and properties of the surface S to estimate the expressions I_1 and I_2. At first, let us divide the inner integral in I_1 by the sum of two integrals by $S_{r_0/2}(x)$ and $S \setminus S_{r_0/2}(x)$and prove that for each $T > 0$ and all $t \in (0,T]$, $x \in S$ the following inequalities

$$|I_1| \leq C_T \int_0^t \tau^{-\beta}((t-\tau)^{-1+\gamma/\alpha} + (t-\tau)^{-1+1/\alpha})d\tau \leq$$
$$\leq C_T \int_0^t \tau^{-\beta}(t-\tau)^{-1+\gamma/\alpha}d\tau = C_T B\left(1-\beta,\frac{\gamma}{\alpha}\right)t^{-\beta+\gamma/\alpha}$$

are true. Therefore, I_1 is finite.

In order to show that the I_2 is finite, let us divide the integral by S in I_2 by the sum of two ones: the fist one by $S_{r_0/2}(x)$ and the second one by $S \setminus S_{r_0/2}(x)$. The second integral is convergent because it can be absolutely estimated from above by the expression

$$C_T \int_0^t \tau^{-\beta}d\tau \int_{S\setminus S_{r_0/2}(x)} \frac{d\sigma_y}{((t-\tau)^{1/\alpha} + |y-x|)^{d+\alpha-1}} \leq$$
$$\leq C_T \int_0^t \tau^{-\beta}(t-\tau)^{-1+1/\alpha}d\tau = C_T B\left(1-\beta,\frac{1}{\alpha}\right)t^{-\beta+1/\alpha}.$$

In order to prove that the first integral convergent we absolutely estimate it above by the expression

$$C_T \int_{S_{r_0/2}(x)} d\sigma_y \int_0^t \tau^{-\beta}d\tau \left|\int_{\mathbb{R}^d}(|\xi|^{\alpha-2}\xi - (Q\xi,\xi)^{\frac{\alpha}{2}-1}Q\xi,\hat{\nu}(x))e^{i(\xi,x-y)-(t-\tau)(Q\xi,\xi)^{\frac{\alpha}{2}}}d\xi\right|.$$

In the inner integral, we use the substitution of the variable ξ by the rule $\xi = \zeta(t-\tau)^{-1/\alpha}$. Obviously, such integral is the Fourier transform calculated at the point $\frac{x-y}{(t-\tau)^{1/\alpha}}$ of the absolutely integrable function (remind that x is fixed) $(e^{-(Q\zeta,\zeta)^{\frac{\alpha}{2}}}(|\zeta|^{\alpha-2}\zeta - (Q\zeta,\zeta)^{\frac{\alpha}{2}-1}Q\zeta, \hat{\nu}(x)))_{\zeta\in\mathbb{R}^d}$, which is monotonically decreasing at the infinity. From the general theory of Fourier integrals (see, for example, [4, §3]) it follows that there exists a constant $M > 0$ such that the absolute value of the integral of the function mentioned above can be estimated above by the expression $M|x-y|^{-\theta}$, where $0 < \theta < 1$. Therefore, our first integral is absolutely estimated above by the expression (here $\rho_0 > 0$ is a constant)

$$C_T \int_0^t \tau^{-\beta} d\tau \int_{S_{r_0/2}(x)} \frac{d\sigma_y}{|y-x|^\theta}(t-\tau)^{-1+\frac{\theta}{\alpha}} \le$$
$$\le C_T \int_0^{\rho_0} \frac{r^{d-2}}{r^\theta} dr \int_0^t \tau^{-\beta}(t-\tau)^{-1+\frac{\theta}{\alpha}} d\tau = C_T \frac{{\rho_0}^{d-1-\theta}}{d-1-\theta} B\left(1-\beta, \frac{\theta}{\alpha}\right) t^{-\beta+\frac{\theta}{\alpha}}.$$

So, I_2 is convergent and, as a consequence, the value $\mathbf{B}^{dv}_{\hat{\nu}(x)} v(t,\cdot)(x)$ is finite. □

Remark. Integral (8) is named by the direct value of the action of the operator $\mathbf{B}_{\hat{\nu}(x)}$ on single layer potential (3) at the point $x \in S$.

The next statement is the jump theorem that has been already mentioned above.

Theorem 2. *Let S be a two-sided surface in $\mathbb{R}^d$ from the class $H^{1+\gamma}$ with some $\gamma \in (0;1)$, which separates the set $\mathbb{R}^d$ into two open sets and for which one of the conditions (A) or (B) is fulfilled. Let a continuous function $(\psi(t,x))_{t\ge0,x\in S}$ satisfy the inequality*

$$|\psi(t,x)| \le K_T t^{-\beta}, \quad 0 < t \le T, x \in S$$

for each $T > 0$ with some constants $\beta < 1$ and $K_T > 0$ (the latter may depend to T). Then for all $t \ge 0$, $x \in S$ the following equality ($\nu(x) = Qn_x$, $\hat{\nu}(x) = Q^{-1}n_x$)

$$\lim_{z\to x\pm} \mathbf{B}_{\hat{\nu}(x)} v(t,\cdot)(z) = \mp\frac{1}{2}\psi(t,x) + \mathbf{B}^{dv}_{\hat{\nu}(x)} v(t,\cdot)(x),$$

holds true, where $z \to x\pm$ means $z = x + \delta\nu(x)$ and $\delta \to 0\pm$.

Proof. Using equality (7), we get the presentation $\mathbf{B}_{\nu(x)} v(t,\cdot)(z) = I_1 + I_2 + I_3 + I_4$, where

$$I_1 = \frac{1}{\alpha}\int_0^t \frac{d\tau}{t-\tau}\int_S (y-x, \hat{\nu}(x)) g(t-\tau, x, y)\psi(\tau,y) d\sigma_y +$$
$$+\frac{i}{(2\pi)^d}\int_0^t d\tau \int_S \psi(\tau,y) d\sigma_y \int_{\mathbb{R}^d} (|\xi|^{\alpha-2}(\xi,\hat{\nu}(x)) - (Q\xi,\xi)^{\frac{\alpha}{2}-1}(Q\xi,\hat{\nu}(x))) e^{i(\xi,x-y)-(t-\tau)(Q\xi,\xi)^{\frac{\alpha}{2}}} d\xi =$$
$$\mathbf{B}^{dv}_{\hat{\nu}(x)} v(t,\cdot)(x),$$
$$I_2 = \frac{1}{\alpha}\int_0^t \frac{d\tau}{t-\tau}\int_S (y-x,\hat{\nu}(x))[g(t-\tau, x+\delta\nu(x), y) - g(t-\tau,x,y)]\psi(\tau,y) d\sigma_y,$$
$$I_3 = -\frac{\delta}{\alpha}\int_0^t \frac{d\tau}{t-\tau}\int_S g(t-\tau, x+\delta\nu(x), y)\psi(\tau,y) d\sigma_y,$$

$$I_4 = \frac{i}{(2\pi)^d} \int_0^t d\tau \int_S \psi(\tau, y) d\sigma_y \int_{\mathbb{R}^d} (e^{i\delta(\xi,\nu(x))} - 1) e^{i(\xi, x-y) - (t-\tau)(Q\xi,\xi)^{\frac{\alpha}{2}}} \times$$
$$\times (|\xi|^{\alpha-2}(\xi, \nu(x)) - (Q\xi, \xi)^{\frac{\alpha}{2}-1}(Q\xi, \nu(x)))\, d\xi.$$

First, consider the expression I_4, submitting it as the sum of integrals J_1 i J_2 of the same function: J_1 with respect to $(\tau, y, \xi) \in (t-\rho, t) \times S \times \mathbb{R}^d$ and J_2 with respect to $(\tau, y, \xi) \in (0, t-\rho) \times S \times \mathbb{R}^d$, where $0 < \rho < t$ is some constant (t is fixed). Since the function $(e^{i\delta(\xi,\nu(x))} - 1)_{\xi \in \mathbb{R}^d}$ is bounded and the integral

$$\int_{t-\rho}^t d\tau \int_S \psi(\tau, y) d\sigma_y \int_{\mathbb{R}^d} e^{i(\xi, x-y) - (t-\tau)(Q\xi,\xi)^{\frac{\alpha}{2}}} (|\xi|^{\alpha-2}(\xi, \hat\nu(x)) - (Q\xi, \xi)^{\frac{\alpha}{2}-1}(Q\xi, \hat\nu(x)))\, d\xi$$

is convergent, the integral J_1 can be made arbitrary small by the choice of ρ .

The integral J_2 is uniformly convergent with respect to δ on each segment $[0; \delta_0]$, $\delta_0 > 0$. Then $J_2 \to 0$ for $\delta \to 0$. Hence, $I_4 \to 0$ for $\delta \to 0$.

Now, investigate the behavior of I_2 for $\delta \to 0$. Get the integral I_2 by the sum of the integrals of the same function over the sets $(0, t-\rho) \times S$, $(t-\rho, t) \times S_{r_0/2}(x)$, and $(t-\rho, t) \times (S \setminus S_{r_0/2}(x))$ with some $0 < \rho < t$ (remind that $t > 0$ is fixed and r_0 is the constant mentioned above in Section 1.4). We denote them by J_1, J_2, J_3, respectively.

Let us estimate each of these terms starting with J_2.

$$|J_2| = \left| \int_{t-\rho}^t \frac{d\tau}{t-\tau} \int_{S_{r_0/2}(x)} (y-x, \hat\nu(x)) \psi(\tau, y)(g(t-\tau, x+\delta\nu(x), y) - g(t-\tau, x, y)) d\sigma_y \right| \le$$
$$\le K_t N_0 \int_{t-\rho}^t \frac{d\tau}{\tau^\beta} \int_{S_{r_0/2}(x)} \frac{|(y-x, \hat\nu(x))| d\sigma_y}{((t-\tau)^{1/\alpha} + |y-x-\delta\nu(x)|)^{d+\alpha}} +$$
$$+ K_t N_0 \int_{t-\rho}^t \frac{d\tau}{\tau^\beta} \int_{S_{r_0/2}(x)} \frac{|(y-x, \hat\nu(x))| d\sigma_y}{((t-\tau)^{1/\alpha} + |y-x|)^{d+\alpha}}.$$

Let $\tilde y$ be the orthogonal projection of $y \in S_{r_0/2}(x)$ on the tangent hyperplane to S at the point x. It is obviously that $|y-x| \ge |\tilde y - x|$. Moreover, the inequalities $0 < \mathrm{const}_1 \le \frac{|y-z|}{|\tilde y - z|} \le \mathrm{const}_2$ are fulfilled for all $y \in S_{r_0/2}(x)$ and $z = x + \zeta n_x$, where $\zeta \in [-|\delta|; |\delta|]$ (see [2, Ch.V, §1]). Therefore, using a local coordinate system with the origin at the point x, and the fact that $|(y-x, \hat\nu(x)| \le \mathrm{const}\, |z|^{1+\gamma}$ (z is the local coordinate of the point $\tilde y$), some non-difficult calculations lead us to the relations $|J_2| \le C_t \left(t^{-\beta+\frac{\gamma}{\alpha}} - (t-\rho)^{-\beta+\frac{\gamma}{\alpha}}\right) \to 0$ ($\rho \to 0+$).

Farther, using inequalities (4) again, we get

$$|J_3| = \left| \int_{t-\rho}^t \frac{d\tau}{t-\tau} \int_{S\setminus S_{r_0/2}(x)} (y-x, \hat\nu(x)) \psi(\tau, y)(g(t-\rho, x+\delta\nu(x), y) - g(t-\tau, x, y)) d\sigma_y \right| \le$$
$$\le C_t N_0 \int_{t-\rho}^t \frac{d\tau}{\tau^\beta} \int_{S\setminus S_{r_0/2}(x)} \frac{|y-x| d\sigma_y}{((t-\tau)^{1/\alpha} + |y-x-\delta\nu(x)|)^{d+\alpha}} +$$
$$+ C_t N_0 \int_{t-\rho}^t \frac{d\tau}{\tau^\beta} \int_{S\setminus S_{r_0/2}(x)} \frac{|y-x| d\sigma_y}{((t-\tau)^{1/\alpha} + |y-x|)^{d+\alpha}}.$$

Let us consider the fulfillment of conditions (A) and (B) separately. Let condition (A) be fulfilled: S is bounded and closed. Then, since $|y-x| \ge \delta_0$ (for constant δ_0 see Section 1.4),

$|y-x-\delta\nu(x)|\geq|y-x|-|\delta||\nu(x)|\geq\delta_0-|\delta||\nu(x)|$ for each $y\in S\setminus S_{r_0/2}(x)$, taking the number δ such that $|\nu(x)||\delta|<\delta_0$, we get $|J_3|\leq C_t\left(t^{1-\beta}-(t-\rho)^{1-\beta}\right)\to 0$ for $\rho\to 0$.

If condition (B) is fulfilled, we write

$$|J_3|\leq C_t\int_{t-\rho}^{t}\frac{d\tau}{t^\beta}\int_{B_r(0)}\frac{|z|dz}{(|z|-|\delta|)^{d+\alpha}}+C_t\int_{t-\rho}^{t}\frac{d\tau}{t^\beta}\int_{B_r(0)}\frac{|z|dz}{(|z|)^{d+\alpha}},$$

where $B_r(0)\subset\mathbb{R}^{d-1}$ is a ball with some radius $r>0$ centered at the origin. Choosing some number $\delta\in(-r,r)$, we get $|J_3|\leq C_t\left(t^{1-\beta}-(t-\rho)^{1-\beta}\right)\to 0$ for $\rho\to 0$.

Now, let us choose and fix a constant $\rho>0$ such as to make the sum J_2 and J_3 small enough and consider J_1. Since the function $g(t,x,y)$ is uniformly continuous on sets of the form $[\rho,+\infty)\times K_1\times K_2$, where K_1 and K_2 are any compacts in $\mathbb{R}^d$, we have $J_1\to 0$ for $\delta\to 0$ when condition (A) is fulfilled. If condition (B) holds, then we use the differentiability with respect to x of the function g. We get

$$\begin{aligned}|g(t-\tau,x+\delta\nu(x),y)-g(t-\tau,x,y)|&=|(\nabla g(t-\tau,\cdot,y)(\theta_{\tau y}\delta\nu(x)+x),\delta\nu(x))|\leq\\&\leq N_1\frac{t-\tau}{((t-\tau)^{1/\alpha}+|y-x|-\theta_{\tau y}|\nu(x)||\delta|)^{d+\alpha+1}}|\nu(x)||\delta|,\end{aligned}$$

where $\theta_{\tau y}\in(0,1)$ is some constant, which depends on τ and y.

So, we have

$$\begin{aligned}|J_1|&\leq N_1K_t|\nu(x)||\delta|\int_0^{t-\rho}\frac{d\tau}{\tau^\beta}\int_S\frac{|y-x|d\sigma_y}{((t-\tau)^{1/\alpha}+|y-x|-\theta_{\tau y}|\nu(x)||\delta|)^{d+\alpha+1}}\leq\\&\leq N_1K_t|\nu(x)||\delta|\int_0^{t-\rho}\frac{d\tau}{\tau^\beta}\int_S\frac{|y-x|d\sigma_y}{(\rho^{1/\alpha}+|y-x|-\theta_{\tau y}|\nu(x)||\delta|)^{d+\alpha+1}}.\end{aligned}$$

Let us choose δ such that $|\nu(x)||\delta|<\frac{1}{2}\rho^{\frac{1}{\alpha}}$. Then $|J_1|\to 0$ for $\delta\to 0$.

Hence, we get that $I_2\to 0$ for $\delta\to 0$.

Now, let us consider I_3 and rewrite it by the sum of the following terms:

$$J_1=-\frac{\delta}{\alpha}\psi(t,x)\int_{t-\rho}^{t}\frac{d\tau}{t-\tau}\int_{S_\varepsilon(x)}g(t-\tau,x+\delta\nu(x),y)d\sigma_y,$$

$$J_2=\frac{\delta}{\alpha}\int_{t-\rho}^{t}\frac{d\tau}{t-\tau}\int_{S_\varepsilon(x)}g(t-\tau,x+\delta\nu(x),y)(\psi(t,x)-\psi(\tau,y))d\sigma_y,$$

$$J_3=-\frac{\delta}{\alpha}\int_0^{t-\rho}\frac{d\tau}{t-\tau}\int_{S_\varepsilon(x)}g(t-\tau,x+\delta\nu(x),y)\psi(\tau,y)d\sigma_y,$$

$$J_4=-\frac{\delta}{\alpha}\int_0^{t}\frac{d\tau}{t-\tau}\int_{S\setminus S_\varepsilon(x)}g(t-\tau,x+\delta\nu(x),y)\psi(\tau,y)d\sigma_y,$$

where $\varepsilon>0$, $0<\rho<t$ are some constants, which we are going to choose.

First, consider the term J_4. We have $|y-x-\delta\nu(x)|\geq\varkappa|\tilde{y}-x|\geq\varkappa\hat{\varepsilon}>0$, where $\tilde{y}$ is the project of $y\in S\setminus S_\varepsilon(x)$ on the tangent hyperplane of S at the point x, $\hat{\varepsilon}>0$ is some constant (it depends on ε), $\varkappa>0$ is a constant which is dependent on the matrix P. Then the inequality

$$|J_4|\leq\frac{|\delta|}{\alpha}K_tN_0\int_0^t\frac{d\tau}{t^\beta}\int_{S\setminus S_\varepsilon(x)}\frac{d\sigma_y}{((t-\tau)^{1/\alpha}+\varkappa|\tilde{y}-x|)^{d+\alpha}}.$$

is valid. If the surface S is bounded (condition (A)), then we get that

$$|J_4| \le \frac{|\delta|}{\alpha} K_t N_0 |S| \hat{\varepsilon}^{-d-\alpha} t^{1+\beta} \to 0, \quad \delta \to 0.$$

If condition (B) is fulfilled, then we have the following (remind that $B_{\hat{\varepsilon}}(0)$ is a ball in $\mathbb{R}^{d-1}$)

$$|J_4| \le \frac{|\delta|}{\alpha} K_t N_0 K \int_0^t \frac{d\tau}{t^\beta} \int_{\overline{B_{\hat{\varepsilon}}(0)}} \frac{dz}{((t-\tau)^{1/\alpha} + \varkappa |z|)^{d+\alpha}} \le$$
$$\le \frac{|\delta|}{\alpha} C_t \int_0^t \frac{d\tau}{t^\beta} \int_{\overline{B_{\hat{\varepsilon}}(0)}} \frac{dz}{|z|^{d+\alpha}} \to 0, \quad \delta \to 0.$$

Here we assume that the function ∇F_x is bounded and $\sqrt{1+|\nabla F_x|^2} \le K$ with some constant $K > 0$.

The next step is the consideration of J_3. We can write the following expressions

$$|J_3| \le \frac{|\delta|}{\alpha} C_t N_0 \int_0^{t-\rho} \frac{d\tau}{t^\beta} \int_{S_\varepsilon(x)} \frac{d\sigma_y}{((t-\tau)^{1/\alpha} + |y - x - \delta\nu(x)|)^{d+\alpha}} \le$$
$$\le \frac{|\delta|}{\alpha} C_t N_0 |S_\varepsilon| \rho^{-\frac{d+\alpha}{\alpha}} (t-\rho)^{1-\beta} \frac{1}{1-\beta} \to 0, \quad \delta \to 0.$$

Farther, we prove the limit existence of J_1 for all fixed ρ and ε.

Let us denote the tangent hyperplane to S at the point $x \in S$ by Π_x and consider the expression

$$R = -\frac{\delta}{\alpha} \int_0^\rho \frac{d\tau}{\tau} \int_{\Pi_x} g(\tau, x + \delta\nu(x), y) d\sigma_y.$$

It can be established (see [1]) that

$$R = -\frac{\delta}{\alpha\pi} \int_0^\rho \frac{d\tau}{\tau} \int_0^\infty e^{-\tau r^\alpha} \cos(\delta r) dr = -\frac{1}{2} \operatorname{sign} \delta + \frac{1}{\pi} \int_0^\infty e^{-\rho r^\alpha} \frac{\sin \delta r}{r} dr \to \mp \frac{1}{2},$$

for $\delta \to \pm 0$.

Let us prove that $\lim\limits_{\delta \to 0} (J_1{}^{(2)} - \psi(t,x) R) = 0$. In order to do this we consider

$$\frac{\delta}{\alpha} \int_0^\rho \frac{d\tau}{\tau} \left(\int_{S_\varepsilon(x)} - \int_{\Pi_x} \right) g(\tau, x + \delta\nu(x), y) d\sigma_y =$$
$$= \frac{\delta}{\alpha} \int_0^\rho \frac{d\tau}{\tau} \left(\int_{S_\varepsilon(x)} - \int_{\Pi_\varepsilon(x)} \right) g(\tau, x + \delta\nu(x), y) d\sigma_y -$$
$$- \frac{\delta}{\alpha} \int_0^\rho \frac{d\tau}{\tau} \int_{\Pi_x \setminus \Pi_\varepsilon(x)} g(\tau, x + \delta\nu(x), y) d\sigma_y = J' + J'',$$

where $\Pi_\varepsilon(x)$ is the set $S_\varepsilon(x)$ projection on Π_x.

Taking into account the properties of the surface S, it is easy to understand that there exists a constant $\theta > 0$ such that for each $y \in \Pi_x \setminus \Pi_\varepsilon(x)$ we have the inequality $|y - x| \ge \theta$. Then, choosing δ such that the inequality $|\delta| < \frac{\theta}{2|\nu(x)|}$ holds, we get (we use spherical coordinates)

$$|J''| \le C|\delta|^{-\alpha} \int_{\theta/|\delta||\nu(x)|}^\infty \frac{r^{d-2} dr}{(r-1)^{d+\alpha}}$$

with some constant $C > 0$. From here, using L'Hôspital's rule, we get that $J'' \to 0$ for $\delta \to 0$.

For the estimating of J' we move to the local coordinate system with the origin at the point x and the vector n_x as the unit vector of the last axis. Then we have

$$S_\varepsilon(x) = \{u \in \mathbb{R}^d : u^d = F_x(u^{<d>}), u^{<d>} \in D_\varepsilon(x) \subset \mathbb{R}^{d-1}\},$$

$$\Pi_\varepsilon(x) = \{u \in \mathbb{R}^d : u^d = 0, u^{<d>} \in D_\varepsilon(x) \subset \mathbb{R}^{d-1}\},$$

where $D_\varepsilon(x)$ is some bounded and closed set, which depends only of the surface S shape, $u^{<d>} = (u_1, u_2, ..., u_{d-1})$. Taking into account estimate (4) it is easy to obtain the inequalities

$$|J'| \le C|\delta|\frac{1}{\alpha}\int_0^\rho \frac{d\tau}{\tau}\int_{D_\varepsilon(x)} \frac{\tau|v|^\gamma(1+|v|^\gamma)dv}{(\tau^{1/\alpha}+k\sqrt{|v|^2+\delta^2})^{d+\alpha}} \le C|\delta|\int_0^\rho d\tau\int_0^{\varepsilon_0}\frac{r^{d-2+\gamma}dr}{(\tau^{1/\alpha}+k\sqrt{r^2+\delta^2})^{d+\alpha}},$$

where $k > 0$, $\varepsilon_0 > 0$ are some constants. By the changing the order of integration and taking into account the equality

$$\int_0^\infty \frac{d\tau}{(\tau^{1/\alpha}+a)^{d+\alpha}} = \alpha B(d,\alpha)a^{-d},$$

which is corrected for all $a > 0$, we get

$$|J'| \le C|\delta|\int_0^{\varepsilon_0}\frac{r^{d-2+\gamma}dr}{(\sqrt{r^2+\delta^2})^d} \le C\int_0^\infty \frac{r^{d-2+\gamma}dr}{(\sqrt{r^2+1})^d}|\delta|^\gamma.$$

Then $J' \to 0$ for $\delta \to 0$.

So, we reach to the conclusion that $J_1 - R\psi(t,x) \to 0$ for $\delta \to 0$. Hence, $\lim_{\delta\to 0\pm} J_1 = \pm\frac{1}{2}\psi(t,x)$. From here, it follows that (since $\psi(t,x)$ is continuous and, consequently, it is uniformly continuous on each compact set) the value $\overline{\lim}_{\delta\to 0}|J_2|$ can be made arbitrary small by the choice of ρ and ε.

Finally, we have the equality

$$\lim_{z\to x\pm} \mathbf{B}_{\hat\nu(x)}v(t,\cdot)(z) = \mp\frac{1}{2}\psi(t,x) + \mathbf{B}^{dv}_{\hat\nu(x)}v(t,\cdot)(x).$$

□

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