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# FUNDAMENTALS OF METRIC THEORY OF REAL NUMBERS IN THEIR $\overline{Q}_3$-REPRESENTATION

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In the paper we study encoding of the fractional part of a real number with an infinite alphabet (set of digits) coinciding with the set of non-negative integers. The geometry of this encoding is generated by $Q_3$-representation of real numbers, which is a generalization of the classical ternary representation. The new representation has infinite alphabet, zero redundant and can be efficiently used to specify mathematical objects with fractal properties.

We have been studied the functions preserving the "tails" of $\overline{Q_3}$-representation of numbers and the set of such functions, some metric problems and some problems of the probability which are connected with $\overline{Q_3}$-representation.

**1. Introduction.** There are three well-known classical constructive theories of real numbers proposed by German mathematicians Weierstrass, Cantor, and Dedekind in the second half of the 19th century. A general axiomatic theory of real numbers was created somewhat later. An interesting theory was proposed, on the level of concepts, by Kolmogorov [6] and was implemented by Kavun [5]. As the Weierstrass theory it needs not a theory of rational numbers. A full appreciation that many such theories can be constructed came relatively recently, namely in the late 20th century.

There are various procedures of coding (representation) of real numbers based on a finite alphabet $\{0, 1, \dots, s-1\} \equiv A_s$, $1 < s \in \mathbb{N}$. For various reasons, two-character encoding deserves special attention. The ternary system is also worth noting as the most economic in a certain sense. The advantages and disadvantages of these systems are discussed in [22].

The *coding of numbers* from $[0, 1]$ by using a (finite or infinite) alphabet $A$ is defined as a surjective mapping

$$\varphi : A \times A \times \ldots \times A \times \ldots \equiv L \to [0, 1].$$

The symbolic notation of the number $x = \varphi((a_n))$, which is the image of the sequence $(a_n)$, in the form $\Delta^{\varphi}_{a_1 a_2 \ldots a_n \ldots}$ is called its *$\varphi$-representation* (or *$\varphi$-code*).

As one of the simplest procedures used to code the numbers $x \in [0, 1]$ by using the alphabet $A_s$, $2 \le s \in \mathbb{N}$, we can mention the $s$-ary representation of a number:

$$x = \frac{\alpha_1}{s} + \frac{\alpha_2}{s^2} + \ldots + \frac{\alpha_n}{s^n} + \ldots \equiv \Delta^{s}_{\alpha_1 \alpha_2 \ldots \alpha_n \ldots}.$$

Geometry of numbers (or geometric number theory) is a branch of the number theory that studies the number-theoretic problems using geometric concepts and methods. It was created

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in the works of Kepler (17th century) and Lagrange (18th century). Gauss (theory of trigonometric sums), Dirichlet (Dirichlet series), Corkin, Zolotarev, and Ye. S. Fedorov made a significant contribution to its formation. Geometry of numbers became an independent branch of mathematics between the 19th and 20th centuries after papers of H. Minkowski and G. Voronoi. Especially, the publication in 1896 of the fundamental monograph "Geometry of numbers" (Geometrie der Zahlen, Leipzig) [9] by H. Minkowski stimulated its development.

In his doctoral dissertation "On one generalization of the algorithm of continued fractions" (1896), defended in 1897, G. Voronoi gave a generalization of continued fractions. An application to numbers depending on a root of an equation of the third degree, the generalization has the periodicity property and can be used to find the basic units and solutions of other problems of the theory. One of the central directions in the development of number geometry is the geometric theory of positive quadratic forms, which includes the problems of lattice packages and spherical coatings.

A separate independent direction of research in the branch of geometry of numbers is the application of geometric methods in the theory of representation of numbers in some system (number system). Such studies are to some extent connected with the ideas of G. Voronoi. Namely, the geometry of different representations of real numbers is still developing. It based on the study of the geometric meaning of digits, metric relations, topologically metric properties of the sets of numbers determined by the conditions imposed on their representations, etc., and the applications to the construction of various mathematical objects and having complex (inhomogeneous) local structures [8]. Let us briefly consider elements of this area of the geometry of numbers.

Let $(c_1, c_2, \ldots, c_m)$ be a fixed ordered collection of elements of the alphabet $A$. A *cylinder of rank $m$ with base $c_1c_2\ldots c_m$ in the coding $\varphi$* is defined as the set $\Delta^{\varphi}_{c_1c_2\ldots c_m}$ all numbers $x \in [0,1]$ admitting the following $\varphi$-representation

$$x = \Delta^{\varphi}_{c_1c_2\ldots c_m\alpha_{m+1}\ldots\alpha_{m+k}\ldots}, \ \ \alpha_{m+i} \in A.$$

The segment $[0,1]$ is called a *cylinder of rank zero* and is denoted by $\Delta$.

This definition directly yields the following properties of the cylinders:

1) $\Delta^{\varphi}_{c_1c_2\ldots c_m} = \bigcup\limits_{i\in A} \Delta^{\varphi}_{c_1c_2\ldots c_m i}$; 2) $\Delta = \bigcup\limits_{i_1\in A}\bigcup\limits_{i_2\in A}\cdots\bigcup\limits_{i_n\in A} \Delta^{\varphi}_{i_1i_2\ldots i_n}$ for any $n$.

In the $s$-ary representation of numbers, the role of cylinders is played by segments, namely,

$$\Delta^{s}_{c_1c_2\ldots c_m} = \Big[\sum_{i=1}^{m} \frac{c_i}{s^i}; \frac{1}{s^m} + \sum_{n=m+1}^{\infty} \frac{c_n}{s^n}\Big].$$

It is known that a coding system has *redundancy zero* [22], if all numbers (or the major part of numbers) possesses the unique representations and only their negligibly small part has two representations.

As examples of coding of the numbers by an infinite alphabet, we can mention the $L-$, $E-$ and $S-$ representations based on the expansions of numbers in the Luroth [7,18,25], Engel [17], Sylvester [11], Sierpiński-Pierce [2,10,21], Ostrogradsky [2,3,12] series of positive terms.

A coding is called *continuous*, if the cylinder is an interval (segment, half-segment, or half-interval) and, moreover, for any sequence $(a_n)$, $a_n \in A$, the section $\bigcap\limits_{m=1}^{\infty} \Delta^{\varphi}_{a_1a_2\ldots a_m} \equiv \Delta^{\varphi}_{a_1a_2\ldots a_m\ldots}$ is a number (point) and, in addition, $\Delta^{\varphi}_{a_1a_2\ldots a_m\ldots} = x \to x' = \Delta^{\varphi}_{a'_1a'_2\ldots a'_m\ldots} \equiv m \to \infty$, where $a_m \neq a'_m$, but $a_i = a'_i$ for $i < m$.

A continuous $\varphi$-representation is called a *Q-representation*, if the alphabet $A$ is finite and, for each $i \in A \equiv A_s$ the following relation is true $\sup \Delta^{Q}_{c_1c_2\ldots c_m i} = \inf \Delta^{Q}_{c_1c_2\ldots c_m[i+1]}$ and the metric ratio $\dfrac{|\Delta^{\varphi}_{c_1c_2\ldots c_m i}|}{|\Delta^{\varphi}_{c_1c_2\ldots c_m}|} \equiv q_i = const$, holds. This ratio is called *main* (this is the most important relation for the metric theory).

The following statement can be regarded as a constructive introduction of $Q$-representations [13, 19]:

For any $x \in [0,1]$ there exists a sequence $(\alpha_n)$, $\alpha_n \in A_s$, such that

$$x = \beta_{\alpha_1} + \sum_{k=2}^{\infty}\left[\beta_{\alpha_k}\prod_{j=1}^{k-1} q_{\alpha_j}\right] = \Delta^{Q}_{\alpha_1\alpha_2\ldots\alpha_n\ldots},$$

where $\beta_0 = 0$, $\beta_i = q_0 + q_1 + \ldots + q_{i-1}$, $0 < i \leq s$. It was used to construct various functions [1, 15, 24], mathematical objects with a complex local structure [19, 20].

The following relations are true for a $Q$-representation:

3) $q_0 + q_1 + \ldots + q_{s-1} = 1$; 4) $|\Delta^{Q}_{c_1c_2\ldots c_m}| = \prod\limits_{i=1}^{m} q_{c_i} \to 0$ $(m \to \infty)$.

The classical $s$-ary representation is a $Q$-representation and satisfies the equality $q_0 = q_1 = \ldots = q_{s-1} = \frac{1}{s}$.

The period of a $Q$-representation of a number (if it exists) is indicated in parentheses. There exist numbers with two different $Q$-representations. These are numbers with period $(0)$ or $(s-1)$, such that $\Delta^{Q}_{c_1\ldots c_{m-1}c_m(0)} = \Delta^{Q}_{c_1\ldots c_{m-1}[c_m-1](s-1)}$. These numbers are called *Q-rational*. The set of these numbers is countable. The other numbers are called $Q$-irrational.

If $s = 3$, then a $Q$-representation is called $Q_3$-representation. Denote by $\alpha_k(x)$ is the $k$th digit (symbol) of the $Q_3$-representation of the number $x$.

**2. Formulation of the problem.** In the works of Pratsiovytyi M. V. [14], Pratsiovytyi M. V. and Isaieva T. M. [16], Goncharenko Ya. V. and Lysenko I. M. [4] the following method to recode a two-character representation of fractional part for real number is considered, namely: the classic binary and $Q_2$–representation of the number. In particular, in [4] it is considered

$$[x] = \Delta^{Q_2}_{\underbrace{1\ldots1}_{a_1}0\underbrace{1\ldots1}_{a_2}0\underbrace{1\ldots1}_{a_3}0\ldots} = \Delta^{q_0^{\infty}}_{a_1a_2a_3\ldots},\ a_n \in \mathbb{Z}_0,$$

by means of the infinite-symbolic alphabet $A = \mathbb{Z}_0$, which is called $q_0^{\infty}$-representation. It is a $Q_2$-representation by the content, but it is an infinite-symbolic representation by the form. This recoding establishes a close relationship between number representation systems and finite and infinite alphabets. Each number $x \in [0; 1)$ has the unique $q_0^{\infty}$-representation $\Delta^{q_0^{\infty}}_{a_1a_2\ldots a_n\ldots}$. $q_0^{\infty}$-representation has $N$-self-similar geometry (while geometry of $Q_2$-representation is self-similar). It expands the arsenal of tools for analytical tasks and research of mathematical objects with structurally complex, in particular, fractal properties.

It is not possible to formally transfer this method to other non-two-character coding of numbers due to various reasons. However, there can be used the idea of encoding numbers by symbols that express the lengths of series of consecutive identical digits.

In this paper, we introduce and investigate a modification of the well-known so-called $Q_3$-representation [13, 23], which is a generalization of the classical ternary. Our main task is to study its "geometry" (by representation geometry we mean the geometric interpretation of symbols, the properties of cylinders and semi-cylinders, as well as various metric relations), the solution of some metric problems and some problems of probability theory.

Representation $\Delta^{\overline{Q_3}}_{a_1a_2...a_n...}$ of a real number

$$x = \Delta^{Q_3}_{\underbrace{0\ldots0}_{a_1}\underbrace{1\ldots1}_{a_2}\underbrace{2\ldots2}_{a_3}\ldots\underbrace{0\ldots0}_{a_{3n-2}}\underbrace{1\ldots1}_{a_{3n-1}}\underbrace{2\ldots2}_{a_{3n}}\ldots}, \quad a_n \in \mathbb{Z}_0, \tag{1}$$

where $a_n$ is the "length" of a series of identical consecutive $Q_3$-digits, is called $\overline{Q_3}$-*representation* of the number $x$.

Obviously, it is only a modification of the $Q_3$-representation, but unlike the latter it does not use a three-character alphabet. It uses an infinite alphabet which is the set of non-negative integers.

From the definition it is clear that in the $\overline{Q_3}$-representation of a real number a row can not have more than two zeros, i.e., $a_na_{n+1}a_{n+2} \neq 000, \ n \in \mathbb{N}$. In addition, even agreeing to use only one of the two representations of ternary-rational numbers, namely with the period of form (1) there are problems with the correctness of this definition. They are associated with both the transition from $Q_3$-representation to $\overline{Q_3}$-representation, and "conversely". The last problem is that not every sequence $(a_n)$ of non-negative integers defines $\overline{Q_3}$-representation $\Delta^{\overline{Q_3}}_{a_1a_2...a_n...}$. For example, how to interpret the record $\Delta^{\overline{Q_3}}_{583(090)}$? Another problem is how to write a number whose ternary representation has a prime period of $(i)$?

These problems are solved by the following agreements.

*In order to provide each number with a single $\overline{Q_3}$-representation, which does not allow ambiguous interpretations (interpretations), we agree:*

a) in the $\overline{Q_3}$-representation of a real number a row can not have more than two zeros, i.e., $a_na_{n+1}a_{n+2} \neq 000, \ n \in \mathbb{N}$;

b) do not use $Q_3$-representation with period of form (1);

c) the number $x = \Delta^{Q_3}_{...(i)}$ is to write $\Delta^{\overline{Q_3}}_{...1001001001...}$, where $i = \{0,1\}$;

d) use two zeros in a row for the $\overline{Q_3}$-representation of the number in exceptional cases, namely: if $x = \Delta^{Q_3}_{2\alpha_2\alpha_3...}$, then $x = \Delta^{\overline{Q_3}}_{00a_3...}$ where $i = \{0,1\}$, or the number has a prime period $(i)$ which was mentioned in (c).

After the introduction of these requirements, any number $x \in [0,1)$ has the unique $\overline{Q_3}$-representation.

**3. Geometry of $\overline{Q_3}$-representation of numbers.** Define the transition from a $Q_3$-representation to the $\overline{Q_3}$-representation, i.e. from the sequence $(\alpha_n)$ to $(a_n)$, where $\alpha_n \in \{0,1,2\}$, $a_n \in \mathbb{Z}_0$. Due to the uniqueness of the $\overline{Q_3}$-representation $a_k(x)$ is a function of a number that determines the length of the $k$th series of identical ternary characters (including zero-length series). Obviously, for all $n \in \mathbb{N}$ there are equalities:

$$a_1 = \begin{cases} 0, \ \alpha_1 \neq 0, \\ n, \ \alpha_{n+1} \neq 0, \ \alpha_j = 0, \ j \leq n; \end{cases} \qquad a_2 = \begin{cases} 0, \ \ \alpha_{a_1+1} = 2, \\ n, \ , \alpha_{a_1+1} = ... = \alpha_{a_1+n} = 1, \ \alpha_{a_1+n+1} \neq 1; \end{cases}$$

$$a_3 = \begin{cases} 0, \ \ \alpha_{a_1+a_2+1} \neq 2, \ a_2 \neq 0, \\ n, \ \alpha_{a_1+a_2+1} = ... = \alpha_{a_1+a_2+n} = 2, \ \alpha_{a_1+a_2+n+1} \neq 2; \end{cases}$$

$$\cdots$$

$$a_{3k-i} = \begin{cases} 0, \ \ \alpha_{t+1} \neq 2-i, \ a_{3k-i-1} \neq 0, \\ n, \ \alpha_{t+1} = ... = \alpha_{t+n} = 2-i, \ \alpha_{t+n+1} \neq 2-i, \end{cases} \qquad \text{where} \ \sum_{j=1}^{3k-i-1} a_j = t, \ i = \{0,1,2\}.$$

The transition from a $\overline{Q_3}$-representation to a $Q_3$-representation (from the sequence of characters $(a_n)$ to the sequence of characters $(\alpha_n)$) is also quite obvious and is determined by the following equations:

$$
\begin{gathered}
\alpha_1 = 0, \;\; \alpha_2 = 0, \dots \alpha_{a_1} = 0, \\
\alpha_{a_1+1} = 1, \;\; \alpha_{a_1+2} = 1, \dots \alpha_{a_1+a_2} = 1, \\
\alpha_{a_1+a_2+1} = 2, \dots \alpha_{a_1+a_2+a_3} = 2, \\
\dots \\
\alpha_{a_1+a_2+\ldots+a_{3k}+1} = 0, \dots, \alpha_{a_1+a_2+\ldots+a_{3k}+a_{3k+1}} = 0, \\
\alpha_{a_1+a_2+\ldots+a_{3k+1}+1} = 1, \dots, \alpha_{a_1+a_2+\ldots+a_{3k+1}+a_{3k+2}} = 1, \\
\alpha_{a_1+a_2+\ldots+a_{3k+2}+1} = 2, \dots, \alpha_{a_1+a_2+\ldots+a_{3k+2}+a_{3k+3}} = 2, \dots.
\end{gathered}
$$

Geometric meaning of digits of a $\overline{Q_3}$-representation of numbers and essence of related positional and metric problems are disclosed by the following important notion.

A cylinder of rank $m$ with base $c_1c_2\dots c_m$ is a set $\Delta^{\overline{Q_3}}_{c_1c_2\ldots c_m}$ of numbers $x \in [0,1)$ having the $\overline{Q_3}$-representation such that $a_i(x) = c_i$, $i = \overline{1,m}$, i.e.,

$$x = \Delta^{\overline{Q_3}}_{c_1c_2\ldots c_m a_{m+1}\ldots a_{m+k}\ldots}, \;\; a_{m+i} \in \mathbb{Z}_0.$$

We denote the interior of the cylinder by $\overline{\nabla}^{\overline{Q_3}}_{c_1c_2\ldots c_k}$ and call it a cylindrical interval, i.e., $\overline{\nabla}^{\overline{Q_3}}_{c_1c_2\ldots c_k} = \operatorname{int} \Delta^{\overline{Q_3}}_{c_1c_2\ldots c_k}$.

**Lemma 1.** *Cylinders have the following properties:*
*1) if $c_m \neq 0$, then* $\Delta^{\overline{Q_3}}_{c_1c_2\ldots c_m} = \Delta^{Q_3}_{\underbrace{0\ldots0}_{c_1}\underbrace{1\ldots1}_{c_2}\ldots\underbrace{\alpha\ldots\alpha}_{c_m}\beta} \cup \Delta^{Q_3}_{\underbrace{0\ldots0}_{c_1}\underbrace{1\ldots1}_{c_2}\ldots\underbrace{\alpha\ldots\alpha}_{c_m}\gamma}$,
*where* $\alpha \equiv m-1 \pmod 3$, $\beta \equiv m \pmod 3$, $\gamma \equiv m+1 \pmod 3$; *if* $c_m = 0$, *then* $\Delta^{\overline{Q_3}}_{c_1c_2\ldots c_{m-1}0} = \Delta^{Q_3}_{\underbrace{0\ldots0}_{c_1}\underbrace{1\ldots1}_{c_2}\ldots\underbrace{\alpha\ldots\alpha}_{c_{m-1}}\gamma}$, *where* $\alpha \equiv m-2 \pmod 3$, $\gamma \equiv m \pmod 3$;
*2) if $c_m \neq 0$, then* $\Delta^{\overline{Q_3}}_{c_1c_2\ldots c_m} = \Delta^{\overline{Q_3}}_{c_1c_2\ldots c_m 1} \cup \Delta^{\overline{Q_3}}_{c_1c_2\ldots c_m 01}$; *if $c_m = 0$, then* $\Delta^{\overline{Q_3}}_{c_1c_2\ldots c_{m-1}0} = \Delta^{\overline{Q_3}}_{c_1c_2\ldots c_{m-1}01}$;
*3)* $\Delta^{\overline{Q_3}}_{c_1c_2\ldots c_m} = \Delta^{\overline{Q_3}}_{s_1s_2\ldots s_k}$ *if and only if all $c_i = s_i$, $m,k \in N$;*
*4)* $\overline{\nabla}^{\overline{Q_3}}_{c_1c_2\ldots c_m} \neq \overline{\nabla}^{\overline{Q_3}}_{s_1s_2\ldots s_k}$ *if at least one of $c_i \neq s_i$, $m,k \in N$;*
*5) for the Lebesgue measure there are equalities:*

*if $c_m \neq 0$, then*

$$
\lambda(\Delta^{\overline{Q_3}}_{c_1c_2\ldots c_m}) =
\begin{cases}
q_0^{\sum\limits_{\substack{i\in(3k-2)\\ i\le m}} c_i} \cdot q_1^{\sum\limits_{\substack{i\in(3k-1)\\ i\le m-2}} c_i} \cdot q_2^{\sum\limits_{\substack{i\in(3k)\\ i\le m-1}} c_i} \cdot (1-q_0), \; m \in (3k-2), \\
q_0^{\sum\limits_{\substack{i\in(3k-2)\\ i\le m-1}} c_i} \cdot q_1^{\sum\limits_{\substack{i\in(3k-1)\\ i\le m}} c_i} \cdot q_2^{\sum\limits_{\substack{i\in(3k)\\ i\le m-2}} c_i} \cdot (1-q_1), \; m \in (3k-1), \\
q_0^{\sum\limits_{\substack{i\in(3k-2)\\ i\le m-2}} c_i} \cdot q_1^{\sum\limits_{\substack{i\in(3k-1)\\ i\le m-1}} c_i} \cdot q_2^{\sum\limits_{\substack{i\in(3k)\\ i\le m}} c_i} \cdot (1-q_2), \; m \in (3k),
\end{cases}
$$

*if $c_m = 0$, then*

$$\lambda(\Delta^{\overline{Q_3}}_{c_1c_2\ldots c_{m-1}0}) = \begin{cases} q_2 \cdot q_0^{\sum\limits_{\substack{i\in(3k-2)\\ i\le m-1}} c_i} \cdot q_1^{\sum\limits_{\substack{i\in(3k-1)\\ i\le m-3}} c_i} \cdot q_2^{\sum\limits_{\substack{i\in(3k)\\ i\le m-2}} c_i}, & m-1\in(3k-2),\\ q_0 \cdot q_0^{\sum\limits_{\substack{i\in(3k-2)\\ i\le m-2}} c_i} \cdot q_1^{\sum\limits_{\substack{i\in(3k-1)\\ i\le m-1}} c_i} \cdot q_2^{\sum\limits_{\substack{i\in(3k)\\ i\le m-3}} c_i}, & m-1\in(3k-1),\\ q_1 \cdot q_0^{\sum\limits_{\substack{i\in(3k-2)\\ i\le m-3}} c_i} \cdot q_1^{\sum\limits_{\substack{i\in(3k-1)\\ i\le m-2}} c_i} \cdot q_2^{\sum\limits_{\substack{i\in(3k)\\ i\le m-1}} c_i}, & m-1\in(3k); \end{cases}$$

*6)* $\lambda(\Delta^{\overline{Q_3}}_{c_1c_2\ldots c_m}) \to 0$ $(m\to\infty)$ *for* $(c_n)\in A^{\infty}$*;*

*7)* $\bigcap\limits_{m=1}^{\infty} \Delta^{\overline{Q_3}}_{c_1c_2\ldots c_m} \equiv \Delta^{\overline{Q_3}}_{c_1c_2\ldots c_m\ldots} = x\in[0,1)$*;*

*8) the main metric ratio has the following forms*

$$\frac{\lambda(\Delta^{\overline{Q_3}}_{c_1c_2\ldots c_m c})}{\lambda(\Delta^{\overline{Q_3}}_{c_1c_2\ldots c_m})} = \frac{q_i^c(1-q_i)}{(1-q_j)} \text{ if } c_m\cdot c\neq 0; \quad \frac{\lambda(\Delta^{\overline{Q_3}}_{c_1c_2\ldots c_m 0})}{\lambda(\Delta^{\overline{Q_3}}_{c_1c_2\ldots c_m})} = \frac{q_{i+1}}{1-q_j} \text{ if } c_m\neq 0,$$

$$\frac{\lambda(\Delta^{\overline{Q_3}}_{c_1c_2\ldots c_{m-1}0c})}{\lambda(\Delta^{\overline{Q_3}}_{c_1c_2\ldots c_{m-1}0})} = q_i^{c-1}(1-q_i),$$

*where* $i\equiv m(\text{mod } 3)$, $j\equiv m-1(\text{mod } 3)$, $m,k\in\mathbb{N}$*;*

*9) the diameter of the cylinder is determined by the formula*

$$\operatorname{diam}(\Delta^{\overline{Q_3}}_{c_1c_2\ldots c_m}) = \begin{cases} q_0^{\sum\limits_{\substack{i\in(3k-2)\\ i\le m}} c_i} \cdot q_1^{\sum\limits_{\substack{i\in(3k-1)\\ i\le m-2}} c_i} \cdot q_2^{\sum\limits_{\substack{i\in(3k)\\ i\le m-1}} c_i} \cdot(1-q_0), & m\in(3k-2),\\ q_0^{\sum\limits_{\substack{i\in(3k-2)\\ i\le m-1}} c_i} \cdot q_1^{\sum\limits_{\substack{i\in(3k-1)\\ i\le m}} c_i} \cdot q_2^{\sum\limits_{\substack{i\in(3k)\\ i\le m-2}} c_i}, & m\in(3k-1),\\ q_0^{\sum\limits_{\substack{i\in(3k-2)\\ i\le m-2}} c_i} \cdot q_1^{\sum\limits_{\substack{i\in(3k-1)\\ i\le m-1}} c_i} \cdot q_2^{\sum\limits_{\substack{i\in(3k)\\ i\le m}} c_i} \cdot(1-q_2), & m\in(3k) \end{cases}$$

*if* $c_m\neq 0$ *and* $c_m=0$, $\operatorname{diam}(\Delta^{\overline{Q_3}}_{c_1c_2\ldots c_{m-1}0}) = \lambda(\Delta^{\overline{Q_3}}_{c_1c_2\ldots c_{m-1}0})$.

*Proof.* Property 1) follows from the definition of a $\overline{Q_3}$-representation and the conditions imposed on it. Namely, if $x\in\Delta^{\overline{Q_3}}_{c_1c_2\ldots c_m}$, then according to equality (1) it follows $x\in\Delta^{Q_3}_{\underbrace{0\ldots0}_{c_1}\underbrace{1\ldots1}_{c_2}\ldots\underbrace{\alpha\ldots\alpha}_{c_m}}$, where $\alpha\equiv m-1$ (mod 3). The $\overline{Q_3}$-representation takes into account the completeness of a series of identical digits. Therefore, if $c_m\neq 0$ and $Q_3$-digit in $(\sum\limits_{i=1}^{m} c_i+1)$th place is different from $\alpha$, then

$$x\in\Delta^{Q_3}_{\underbrace{0\ldots0}_{c_1}\underbrace{1\ldots1}_{c_2}\ldots\underbrace{\alpha\ldots\alpha}_{c_m}\beta} \text{ and } x\in\Delta^{Q_3}_{\underbrace{0\ldots0}_{c_1}\underbrace{1\ldots1}_{c_2}\ldots\underbrace{\alpha\ldots\alpha}_{c_m}\gamma},$$

where $\alpha+1\equiv\beta$ (mod 3) and $\alpha+2\equiv\gamma$ (mod 3).

If $c_m=0$, then $x\in\Delta^{Q_3}_{\underbrace{0\ldots0}_{c_1}\underbrace{1\ldots1}_{c_2}\ldots\underbrace{\alpha\ldots\alpha}_{c_{m-1}}\underbrace{\beta\ldots\beta}_{c_m=0}\gamma} \equiv \Delta^{Q_3}_{\underbrace{0\ldots0}_{c_1}\underbrace{1\ldots1}_{c_2}\ldots\underbrace{\alpha\ldots\alpha}_{c_{m-1}}\gamma}$, where $\alpha\neq\gamma\neq\beta$, $\alpha\equiv m-2$ (mod 3) and $\gamma\equiv m$ (mod 3).

Property 2) follows from property 1). If $x\in\Delta^{\overline{Q_3}}_{c_1c_2\ldots c_m}$ and $c_m\neq 0$, then

$$x\in\Delta^{Q_3}_{\underbrace{0\ldots0}_{c_1}\underbrace{1\ldots1}_{c_2}\ldots\underbrace{\alpha\ldots\alpha}_{c_m}\beta} \equiv \Delta^{\overline{Q_3}}_{c_1c_2\ldots c_m1} \text{ and } x\in\Delta^{Q_3}_{\underbrace{0\ldots0}_{c_1}\underbrace{1\ldots1}_{c_2}\ldots\underbrace{\alpha\ldots\alpha}_{c_m}\gamma} \equiv \Delta^{\overline{Q_3}}_{c_1c_2\ldots c_m01},$$

i.e., $\Delta^{\overline{Q_3}}_{c_1c_2\ldots c_m} = \Delta^{\overline{Q_3}}_{c_1c_2\ldots c_m1} \cup \Delta^{\overline{Q_3}}_{c_1c_2\ldots c_m01}$.

If $x \in \Delta^{\overline{Q_3}}_{c_1c_2\ldots c_m}$ and $c_m = 0$, then $x \in \Delta^{Q_3}_{\underbrace{0\ldots0}_{c_1}\underbrace{1\ldots1}_{c_2}\ldots\underbrace{\alpha\ldots\alpha}_{c_{m-1}}\gamma} \equiv \Delta^{\overline{Q_3}}_{c_1c_2\ldots c_m01}$, where $\alpha + 2 \equiv \gamma \pmod 3$.

Properties 3) and 4) follow from the uniqueness of the $\overline{Q_3}$-representation of an arbitrary real number $x \in [0,1)$.

Consider property 5). Let $c_m \neq 0$, then according to property 1) we have

$$\Delta^{\overline{Q_3}}_{c_1c_2\ldots c_m} = \Delta^{Q_3}_{\underbrace{0\ldots0}_{c_1}\underbrace{1\ldots1}_{c_2}\ldots\underbrace{\alpha\ldots\alpha}_{c_m}\beta} \cup \Delta^{Q_3}_{\underbrace{0\ldots0}_{c_1}\underbrace{1\ldots1}_{c_2}\ldots\underbrace{\alpha\ldots\alpha}_{c_m}\gamma},$$

where $\alpha \equiv m-1 \pmod 3$, $\beta \equiv m \pmod 3$, $\gamma \equiv m+1 \pmod 3$, and $\Delta^{\overline{Q_3}}_{c_1c_2\ldots c_{m-1}0} = \Delta^{Q_3}_{\underbrace{0\ldots0}_{c_1}\underbrace{1\ldots1}_{c_2}\ldots\underbrace{\alpha\ldots\alpha}_{c_{m-1}}\gamma}$, where $\alpha \equiv m-2 \pmod 3$, $\gamma \equiv m \pmod 3$.

If $m \in (3k-2)$, then for Lebesgue measure there is

$$\lambda(\Delta^{\overline{Q_3}}_{c_1c_2\ldots c_m}) = \lambda(\Delta^{Q_3}_{\underbrace{0\ldots0}_{c_1}\underbrace{1\ldots1}_{c_2}\ldots\underbrace{0\ldots0}_{c_m}1}) + \lambda(\Delta^{Q_3}_{\underbrace{0\ldots0}_{c_1}\underbrace{1\ldots1}_{c_2}\ldots\underbrace{0\ldots0}_{c_m}2}) =$$

$$= q_1 \cdot q_0^{\sum\limits_{\substack{i\in(3k-2)\\ i\leq m}} c_i} \cdot q_1^{\sum\limits_{\substack{i\in(3k-1)\\ i\leq m-2}} c_i} \cdot q_2^{\sum\limits_{\substack{i\in(3k)\\ i\leq m-1}} c_i} + q_2 \cdot q_0^{\sum\limits_{\substack{i\in(3k-2)\\ i\leq m}} c_i} \cdot q_1^{\sum\limits_{\substack{i\in(3k-1)\\ i\leq m-2}} c_i} \cdot q_2^{\sum\limits_{\substack{i\in(3k)\\ i\leq m-1}} c_i} =$$

$$= (1-q_0)q_0^{\sum\limits_{\substack{i\in(3k-2)\\ i\leq m}} c_i} \cdot q_1^{\sum\limits_{\substack{i\in(3k-1)\\ i\leq m-2}} c_i} \cdot q_2^{\sum\limits_{\substack{i\in(3k)\\ i\leq m-1}} c_i}.$$

If $m \in (3k-1)$, then

$$\lambda(\Delta^{\overline{Q_3}}_{c_1c_2\ldots c_m}) = \lambda(\Delta^{Q_3}_{\underbrace{0\ldots0}_{c_1}\underbrace{1\ldots1}_{c_2}\ldots\underbrace{1\ldots1}_{c_m}0}) + \lambda(\Delta^{Q_3}_{\underbrace{0\ldots0}_{c_1}\underbrace{1\ldots1}_{c_2}\ldots\underbrace{1\ldots1}_{c_m}2}) =$$

$$= (q_0+q_2) \cdot q_0^{\sum\limits_{\substack{i\in(3k-2)\\ i\leq m-1}} c_i} \cdot q_1^{\sum\limits_{\substack{i\in(3k-1)\\ i\leq m}} c_i} \cdot q_2^{\sum\limits_{\substack{i\in(3k)\\ i\leq m-2}} c_i} = (1-q_1)q_0^{\sum\limits_{\substack{i\in(3k-2)\\ i\leq m-1}} c_i} \cdot q_1^{\sum\limits_{\substack{i\in(3k-1)\\ i\leq m}} c_i} \cdot q_2^{\sum\limits_{\substack{i\in(3k)\\ i\leq m-2}} c_i}.$$

If $m \in (3k)$, then

$$\lambda(\Delta^{\overline{Q_3}}_{c_1c_2\ldots c_m}) = \lambda(\Delta^{Q_3}_{\underbrace{0\ldots0}_{c_1}\underbrace{1\ldots1}_{c_2}\ldots\underbrace{2\ldots2}_{c_m}0}) + \lambda(\Delta^{Q_3}_{\underbrace{0\ldots0}_{c_1}\underbrace{1\ldots1}_{c_2}\ldots\underbrace{2\ldots2}_{c_m}1}) =$$

$$= (q_0+q_1) \cdot q_0^{\sum\limits_{\substack{i\in(3k-2)\\ i\leq m-2}} c_i} \cdot q_1^{\sum\limits_{\substack{i\in(3k-1)\\ i\leq m-1}} c_i} \cdot q_2^{\sum\limits_{\substack{i\in(3k)\\ i\leq m}} c_i} = (1-q_2)q_0^{\sum\limits_{\substack{i\in(3k-2)\\ i\leq m-2}} c_i} \cdot q_1^{\sum\limits_{\substack{i\in(3k-1)\\ i\leq m-1}} c_i} \cdot q_2^{\sum\limits_{\substack{i\in(3k)\\ i\leq m}} c_i}.$$

Let $c_m = 0$. If $m-1 \in (3k-2)$, then for Lebesgue measure there is

$$\lambda(\Delta^{\overline{Q_3}}_{c_1c_2\ldots c_{m-1}0}) = \lambda(\Delta^{Q_3}_{\underbrace{0\ldots0}_{c_1}\underbrace{1\ldots1}_{c_2}\ldots\underbrace{0\ldots0}_{c_{m-1}}2}) = q_2 \cdot q_0^{\sum\limits_{\substack{i\in(3k-2)\\ i\leq m-1}} c_i} \cdot q_1^{\sum\limits_{\substack{i\in(3k-1)\\ i\leq m-3}} c_i} \cdot q_2^{\sum\limits_{\substack{i\in(3k)\\ i\leq m-2}} c_i}.$$

If $m-1 \in (3k-1)$ or $m-1 \in (3k)$, then

$$\lambda(\Delta^{\overline{Q_3}}_{c_1c_2\ldots c_{m-1}0}) = \lambda(\Delta^{Q_3}_{\underbrace{0\ldots0}_{c_1}\underbrace{1\ldots1}_{c_2}\ldots\underbrace{1\ldots1}_{c_{m-1}}0}) = q_0 \cdot q_0^{\sum\limits_{\substack{i\in(3k-2)\\ i\leq m-2}} c_i} \cdot q_1^{\sum\limits_{\substack{i\in(3k-1)\\ i\leq m-1}} c_i} \cdot q_2^{\sum\limits_{\substack{i\in(3k)\\ i\leq m-3}} c_i},$$

$$\lambda(\Delta^{\overline{Q_3}}_{c_1c_2\ldots c_{m-1}0}) = \lambda(\Delta^{Q_3}_{\underbrace{0\ldots0}_{c_1}\underbrace{1\ldots1}_{c_2}\ldots\underbrace{2\ldots2}_{c_{m-1}}1}) = q_1 \cdot q_0^{\sum\limits_{\substack{i\in(3k-2)\\ i\le m-3}} c_i} \cdot q_1^{\sum\limits_{\substack{i\in(3k-1)\\ i\le m-2}} c_i} \cdot q_2^{\sum\limits_{\substack{i\in(3k)\\ i\le m-1}} c_i}.$$

Property 6) follows from 5) if $m\to\infty$.

A cylinder of rank $m$ with base $c_1c_2\ldots c_m$ is a sequence of nested compacts, and their intersection defines the unique number $x\in[0,1)$. This proves property 7).

Consider property 8). Let $c_m\ne0\ne c$. If $m\in(3k-2)$, then in view of property 5) we have

$$\frac{\lambda(\Delta^{\overline{Q_3}}_{c_1c_2\ldots c_mc})}{\lambda(\Delta^{\overline{Q_3}}_{c_1c_2\ldots c_m})} = \frac{(1-q_1)q_1^c \cdot q_0^{\sum\limits_{\substack{i\in(3k-2)\\ i\le m}} c_i} \cdot q_1^{\sum\limits_{\substack{i\in(3k-1)\\ i\le m-2}} c_i} \cdot q_2^{\sum\limits_{\substack{i\in(3k)\\ i\le m-1}} c_i}}{(1-q_0)q_0^{\sum\limits_{\substack{i\in(3k-2)\\ i\le m}} c_i} \cdot q_1^{\sum\limits_{\substack{i\in(3k-1)\\ i\le m-2}} c_i} \cdot q_2^{\sum\limits_{\substack{i\in(3k)\\ i\le m-1}} c_i}} = \frac{q_1^c(1-q_1)}{(1-q_0)}.$$

If $m\in(3k-1)$, then

$$\frac{\lambda(\Delta^{\overline{Q_3}}_{c_1c_2\ldots c_mc})}{\lambda(\Delta^{\overline{Q_3}}_{c_1c_2\ldots c_m})} = \frac{(1-q_2)q_2^c \cdot q_0^{\sum\limits_{\substack{i\in(3k-2)\\ i\le m-1}} c_i} \cdot q_1^{\sum\limits_{\substack{i\in(3k-1)\\ i\le m}} c_i} \cdot q_2^{\sum\limits_{\substack{i\in(3k)\\ i\le m-2}} c_i}}{(1-q_1)q_0^{\sum\limits_{\substack{i\in(3k-2)\\ i\le m-1}} c_i} \cdot q_1^{\sum\limits_{\substack{i\in(3k-1)\\ i\le m}} c_i} \cdot q_2^{\sum\limits_{\substack{i\in(3k)\\ i\le m-2}} c_i}} = \frac{q_2^c(1-q_2)}{(1-q_1)}.$$

If $m\in(3k)$, then

$$\frac{\lambda(\Delta^{\overline{Q_3}}_{c_1c_2\ldots c_mc})}{\lambda(\Delta^{\overline{Q_3}}_{c_1c_2\ldots c_m})} = \frac{(1-q_0)q_0^c \cdot q_0^{\sum\limits_{\substack{i\in(3k-2)\\ i\le m-2}} c_i} \cdot q_1^{\sum\limits_{\substack{i\in(3k-1)\\ i\le m-1}} c_i} \cdot q_2^{\sum\limits_{\substack{i\in(3k)\\ i\le m}} c_i}}{(1-q_2)q_0^{\sum\limits_{\substack{i\in(3k-2)\\ i\le m-2}} c_i} \cdot q_1^{\sum\limits_{\substack{i\in(3k-1)\\ i\le m-1}} c_i} \cdot q_2^{\sum\limits_{\substack{i\in(3k)\\ i\le m}} c_i}} = \frac{q_0^c(1-q_0)}{(1-q_2)}.$$

Let $c=0$ and $c_m\ne0$. If $m\in(3k-2)$, then

$$\frac{\lambda(\Delta^{\overline{Q_3}}_{c_1c_2\ldots c_mc})}{\lambda(\Delta^{\overline{Q_3}}_{c_1c_2\ldots c_m})} = \frac{q_2 \cdot q_0^{\sum\limits_{\substack{i\in(3k-2)\\ i\le m}} c_i} \cdot q_1^{\sum\limits_{\substack{i\in(3k-1)\\ i\le m-2}} c_i} \cdot q_2^{\sum\limits_{\substack{i\in(3k)\\ i\le m-1}} c_i}}{(1-q_0)q_0^{\sum\limits_{\substack{i\in(3k-2)\\ i\le m}} c_i} \cdot q_1^{\sum\limits_{\substack{i\in(3k-1)\\ i\le m-2}} c_i} \cdot q_2^{\sum\limits_{\substack{i\in(3k)\\ i\le m-1}} c_i}} = \frac{q_2}{(1-q_0)}.$$

If $m\in(3k-1)$, then

$$\frac{\lambda(\Delta^{\overline{Q_3}}_{c_1c_2\ldots c_mc})}{\lambda(\Delta^{\overline{Q_3}}_{c_1c_2\ldots c_m})} = \frac{q_0 \cdot q_0^{\sum\limits_{\substack{i\in(3k-2)\\ i\le m-1}} c_i} \cdot q_1^{\sum\limits_{\substack{i\in(3k-1)\\ i\le m}} c_i} \cdot q_2^{\sum\limits_{\substack{i\in(3k)\\ i\le m-2}} c_i}}{(1-q_1)q_0^{\sum\limits_{\substack{i\in(3k-2)\\ i\le m-1}} c_i} \cdot q_1^{\sum\limits_{\substack{i\in(3k-1)\\ i\le m}} c_i} \cdot q_2^{\sum\limits_{\substack{i\in(3k)\\ i\le m-2}} c_i}} = \frac{q_0}{(1-q_1)}.$$

If $m\in(3k)$, then

$$\frac{\lambda(\Delta^{\overline{Q_3}}_{c_1c_2\ldots c_mc})}{\lambda(\Delta^{\overline{Q_3}}_{c_1c_2\ldots c_m})} = \frac{q_1 \cdot q_0^{\sum\limits_{\substack{i\in(3k-2)\\ i\le m-2}} c_i} \cdot q_1^{\sum\limits_{\substack{i\in(3k-1)\\ i\le m-1}} c_i} \cdot q_2^{\sum\limits_{\substack{i\in(3k)\\ i\le m}} c_i}}{(1-q_2)q_0^{\sum\limits_{\substack{i\in(3k-2)\\ i\le m-2}} c_i} \cdot q_1^{\sum\limits_{\substack{i\in(3k-1)\\ i\le m-1}} c_i} \cdot q_2^{\sum\limits_{\substack{i\in(3k)\\ i\le m}} c_i}} = \frac{q_1}{(1-q_2)}.$$

Let $c_m=0$ and $c\ne0$. If $m\in(3k-2)$, then

$$\frac{\lambda(\Delta^{\overline{Q_3}}_{c_1c_2\ldots c_m c})}{\lambda(\Delta^{\overline{Q_3}}_{c_1c_2\ldots c_m})} = \frac{(1-q_1)q_1^c\cdot q_0^{\sum\limits_{\substack{i\in(3k-2)\\ i\le m}} c_i}\cdot q_1^{\sum\limits_{\substack{i\in(3k-1)\\ i\le m-2}} c_i}\cdot q_2^{\sum\limits_{\substack{i\in(3k)\\ i\le m-1}} c_i}}{q_1q_0^{\sum\limits_{\substack{i\in(3k-2)\\ i\le m-3}} c_i}\cdot q_1^{\sum\limits_{\substack{i\in(3k-1)\\ i\le m-2}} c_i}\cdot q_2^{\sum\limits_{\substack{i\in(3k)\\ i\le m-1}} c_i}} = \frac{q_1^c(1-q_1)}{q_1}.$$

If $m\in(3k-1)$, then

$$\frac{\lambda(\Delta^{\overline{Q_3}}_{c_1c_2\ldots c_m c})}{\lambda(\Delta^{\overline{Q_3}}_{c_1c_2\ldots c_m})} = \frac{(1-q_2)q_2^c\cdot q_0^{\sum\limits_{\substack{i\in(3k-2)\\ i\le m-1}} c_i}\cdot q_1^{\sum\limits_{\substack{i\in(3k-1)\\ i\le m}} c_i}\cdot q_2^{\sum\limits_{\substack{i\in(3k)\\ i\le m-2}} c_i}}{q_2q_0^{\sum\limits_{\substack{i\in(3k-2)\\ i\le m-1}} c_i}\cdot q_1^{\sum\limits_{\substack{i\in(3k-1)\\ i\le m-3}} c_i}\cdot q_2^{\sum\limits_{\substack{i\in(3k)\\ i\le m-2}} c_i}} = \frac{q_2^c(1-q_2)}{q_2}.$$

If $m\in(3k)$, then

$$\frac{\lambda(\Delta^{\overline{Q_3}}_{c_1c_2\ldots c_m c})}{\lambda(\Delta^{\overline{Q_3}}_{c_1c_2\ldots c_m})} = \frac{(1-q_0)q_0^c\cdot q_0^{\sum\limits_{\substack{i\in(3k-2)\\ i\le m-2}} c_i}\cdot q_1^{\sum\limits_{\substack{i\in(3k-1)\\ i\le m-1}} c_i}\cdot q_2^{\sum\limits_{\substack{i\in(3k)\\ i\le m}} c_i}}{q_0q_0^{\sum\limits_{\substack{i\in(3k-2)\\ i\le m-2}} c_i}\cdot q_1^{\sum\limits_{\substack{i\in(3k-1)\\ i\le m-1}} c_i}\cdot q_2^{\sum\limits_{\substack{i\in(3k)\\ i\le m-3}} c_i}} = \frac{q_0^c(1-q_0)}{q_0}.$$

Property 9) follows from properties 1) and 5). □

Let $\Delta^{k_1}_{c_1}$ be the set of all numbers that have a fixed digit $c_1$ in the $k_1$th place of the modified $\overline{Q_3}$-expanding, i.e., $\Delta^{k_1}_{c_1}\equiv\{x: a_{k_1}(x)=c_1\}$.

If $k_1=1$, then the set $\Delta^{k_1}_{c_1}$ is a cylinder $\Delta^{\overline{Q_3}}_{c_1}$.

The set $\Delta^{k_1}_{c_1}$ of numbers from $[0,1)$, for which the $k_1$st digit has a specific value $c_1$, is a union of cylindrical sets: $\Delta^{k_1}_{c_1} = \bigcup\limits_{\substack{(a_1,\ldots,a_{k_1-1})\\ a_i\in\mathbb{Z}_0}} \Delta^{\overline{Q_3}}_{a_1\ldots a_{k_1-1}c_1}$.

Then the Lebesgue measure is defined as the sum of the lengths of these cylinders, i.e.,

$$\lambda(\Delta^{k_1}_{c_1}) = \sum_{\substack{(a_1,\ldots,a_{k_1-1})\\ a_i\in\mathbb{Z}_0}} \lambda\left(\Delta^{\overline{Q_3}}_{a_1\ldots a_{k_1-1}c_1}\right).$$

**Lemma 2.** *For the Lebesgue measure of the set $\Delta^{k_1}_{c_1}$ we have*

$$\lambda(\Delta^{k_1}_{c_1}) = \frac{q_{\alpha_r}^{c_1}(1-q_{\alpha_r})^{\psi(c_1)}}{q_{\alpha_t}^{\psi(c_1)-1}} \sum_{\substack{(a_1,\ldots,a_{k_1-1})\\ a_i\in\mathbb{Z}_0\\ 1\le i\le k_1-1}} q_0^{\sum\limits_{i\in(3n-2)} a_i}\cdot q_1^{\sum\limits_{i\in(3n-1)} a_i}\cdot q_2^{\sum\limits_{i\in(3n)} a_i}, \tag{2}$$

*where* $\alpha_r\equiv k_1-1 \pmod 3$, $\alpha_t\equiv k_1 \pmod 3$ *and* $\psi(c_1)=\begin{cases}1 & \text{if } c_1\ne 0,\\ 0 & \text{if } c_1=0.\end{cases}$

*Proof.* It is clear that $\lambda(\Delta^{k_1}_{c_1}) = \sum\limits_{\substack{(a_1,\ldots,a_{k_1-1})\\ a_i\in\mathbb{Z}_0\\ 1\le i\le k_1-1}} \lambda(\Delta^{\overline{Q_3}}_{a_1a_2\ldots a_{k_1-1}c_1})$. In view of property 5) in Lemma 1, it is obvious that $\lambda(\Delta^{k_1}_{c_1}) = q_{\alpha_r}^{c_1}(1-q_{\alpha_r}) \sum\limits_{\substack{(a_1,\ldots,a_{k_1-1})\\ a_i\in\mathbb{Z}_0\\ 1\le i\le k_1-1}} q_0^{\sum\limits_{i\in(3n-2)} a_i}\cdot q_1^{\sum\limits_{i\in(3n-1)} a_i}\cdot q_2^{\sum\limits_{i\in(3n)} a_i}$, where $\alpha_r\equiv k_1-1 \pmod 3$ for $c_1\ne 0$.

In view of property 5) in Lemma 1, we have

$$\lambda(\Delta^{k_1}_{c_1}) = q_{\alpha_t} \sum_{\substack{(a_1,\dots,a_{k_1-1})\\ a_i\in\mathbb{Z}_0\\ 1\le i\le k_1-1}} q_0^{\sum\limits_{i\in(3n-2)} a_i} \cdot q_1^{\sum\limits_{i\in(3n-1)} a_i} \cdot q_2^{\sum\limits_{i\in(3n)} a_i},$$

where $\alpha_t \equiv k_1 \pmod 3$ for $c_1 = 0$.

Combining these two cases, we obtain equality (2). □

We summarize the results and consider the set $\Delta^{k_1k_2\dots k_m}_{c_1c_2\dots c_m}$.

The set of $\Delta^{k_1k_2\dots k_m}_{c_1c_2\dots c_m} = \{x : a_{k_i}(x) = c_i, i = \overline{1,m}\}$ is called a *semi-cylinder* with a base $\begin{pmatrix} k_1k_2\dots k_m \\ c_1c_2\dots c_m \end{pmatrix}$.

**Theorem 1.** *The Lebesgue measure of the semi-cylinder $\Delta^{k_1k_2\dots k_m}_{c_1c_2\dots c_m}$ is calculated by the formula:*

$$\lambda(\Delta^{k_1k_2\dots k_m}_{c_1c_2\dots c_m}) = \frac{q^{c_m}_{\alpha_r}(1-q_{\alpha_r})^{\psi(c_m)}}{q^{\psi(c_m)-1}_{\alpha_t}} \sum_{\substack{(a_1,\dots,a_{k_1-1},a_{k_1+1},\dots,a_{k_m-1})\\ a_i\in\mathbb{Z}_0\\ 1\le i\le k_m-1\\ i\ne k_1, i\ne k_2,\dots,i\ne k_{m-1}}} q_0^{\sum\limits_{i\in(3n-2)} a_i} \cdot q_1^{\sum\limits_{i\in(3n-1)} a_i} \cdot q_2^{\sum\limits_{i\in(3n)} a_i}, \quad (3)$$

*where* $\alpha_r \equiv k_m - 1 \pmod 3$, $\alpha_t \equiv k_m \pmod 3$ *and* $\psi(c_m) = \begin{cases} 1 & \textit{if } c_m \in \mathbb{N}, \\ 0 & \textit{if } c_m = 0. \end{cases}$

*Proof.* It is clear that

$$\Delta^{k_1k_2\dots k_m}_{c_1c_2\dots c_m} = \bigcup_{\substack{(a_1,\dots,a_{k_1-1},\dots,a_{k_{j-1}+1},\dots,a_{k_j-1},\dots,a_{k_{m-1}+1},\dots,a_{k_m-1})\\ a_i\in\mathbb{Z}_0\\ 1\le i\le k_m-1\\ i\ne k_1, i\ne k_2,\dots,i\ne k_m}} \Delta^{\overline{Q_3}}_{a_1\dots a_{k_1-1}c_1a_{k_1+1}\dots a_{k_m-1}c_m},$$

so
$$\lambda(\Delta^{k_1k_2\dots k_m}_{c_1c_2\dots c_m}) = \sum_{\substack{(a_1,\dots,a_{k_1-1},\dots,a_{k_{j-1}+1},\dots,a_{k_j-1},\dots,a_{k_{m-1}+1},\dots,a_{k_m-1})\\ a_i\in\mathbb{Z}_0\\ 1\le i\le k_m-1\\ i\ne k_1, i\ne k_2,\dots,i\ne k_m}} \lambda(\Delta^{\overline{Q_3}}_{a_1\dots a_{k_1-1}c_1a_{k_1+1}\dots a_{k_m-1}c_m}).$$

In view of property 5) in Lemma 1, it is obvious that (for $c_m \ne 0$)

$$\lambda(\Delta^{k_1k_2\dots k_m}_{c_1c_2\dots c_m}) = q^{c_m}_{\alpha_r}(1-q_{\alpha_r}) \sum_{\substack{(a_1,\dots,a_{k_1-1},a_{k_1+1},\dots,a_{k_m-1})\\ a_i\in\mathbb{Z}_0\\ 1\le i\le k_m-1\\ i\ne k_1, i\ne k_2,\dots,i\ne k_{m-1}}} q_0^{\sum\limits_{i\in(3n-2)} a_i} \cdot q_1^{\sum\limits_{i\in(3n-1)} a_i} \cdot q_2^{\sum\limits_{i\in(3n)} a_i}$$

If $c_m = 0$, then

$$\lambda(\Delta^{k_1k_2\dots k_m}_{c_1c_2\dots c_m}) = q_{\alpha_t} \sum_{\substack{(a_1,\dots,a_{k_1-1},a_{k_1+1},\dots,a_{k_m-1})\\ a_i\in\mathbb{Z}_0\\ 1\le i\le k_m-1\\ i\ne k_1, i\ne k_2,\dots,i\ne k_{m-1}}} q_0^{\sum\limits_{i\in(3n-2)} a_i} \cdot q_1^{\sum\limits_{i\in(3n-1)} a_i} \cdot q_2^{\sum\limits_{i\in(3n)} a_i},$$

where $\alpha_r \equiv k_m - 1 \pmod 3$ and $\alpha_t \equiv k_m \pmod 3$.

Combining the above we get equality (3). □

**4. Tail sets and operators on digits.** Let $Z^{\overline{Q_3}}_{[0;1)}$ be the set of all $\overline{Q_3}$-representations of numbers from $[0;1)$. We introduce binary relation *"has the same tail"* ( symbolically: $\sim$ ) on the set $Z^{\overline{Q_3}}_{[0;1)}$.

Two $\overline{Q_3}$-representations $\Delta^{\overline{Q_3}}_{a_1a_2\ldots a_n\ldots}$ and $\Delta^{\overline{Q_3}}_{b_1b_2\ldots b_n\ldots}$ are said *to have the same tail* (or they *are $\sim$-related*) if there exist positive integers $m$ and $k$ such that $a_{m+j}=b_{k+j}$ for any $j\in\mathbb{N}$.

It is evident that binary relation $\sim$ is an equivalence relation (i.e., it is reflexive, symmetric and transitive) and provides a partition of the set $Z^{\overline{Q_3}}_{[0;1)}$ into equivalence classes. Any equivalence class is said to be a *tail set*. Any tail set is uniquely determined by its arbitrary element (representative).

We say that *two numbers $x$ and $y$ have the same tail* of $\overline{Q_3}$-representation (or they *are $\sim$-related*) if their $\overline{Q_3}$-representations are $\sim$-related. We denote this symbolically as $x\sim y$.

**Theorem 2.** *Any tail set is countable and dense in* $[0;1)$.

*Proof.* Suppose that $K$ is an arbitrary equivalence class, $x_0=\Delta^{\overline{Q_3}}_{c_1c_2\ldots c_k\ldots}$ is its representative. Then it is evident that, for any $m$, there exists a set

$$K_m=\{x: x=\Delta^{\overline{Q_3}}_{a_1\ldots a_kc_{m+1}c_{m+2}\ldots}, a_i\in\mathbb{N}, k=1,2,...\}$$

of numbers $x$ such that $a_{k+j}(x)=c_{m+j}$ for some $k\in\mathbb{N}$ and for any $j\in\mathbb{N}$, and $K=\bigcup\limits_{m\in\mathbb{N}}K_m$.

The set $K$ is countable because it is a countable union of countable sets.

Now we prove that $K$ is a dense set in $[0;1)$. Since number $x$ belongs or does not belong to the set $K$ irrespective of any finite amount of first digits of its $\overline{Q_3}$-representation, we have that any cylinder of arbitrary rank $m$ contains point belonging to $K$. Thus $K$ is an everywhere dense in the half-interval $[0;1)$ set. □

**Theorem 3.** *Quotient set $G\equiv[0;1)/\sim$ is a continuum set.*

*Proof.* To prove that quotient set $G\equiv[0;1)/\sim$ is continuum set, we assume the contrary. Suppose that $G$ is a countable set. Then half-interval $[0;1)$ is a countable set as a countable union of countable sets (equivalence classes of quotient set $G$), according to the Theorem 2. This contradiction proves the theorem. □

Remark that it is easy to introduce a distance function (metric) in the quotient set $G$.

Suppose that function $f$ is defined on the set $Z^{\overline{Q_3}}_{[0;1)}$ and takes values from this set. We say that function $f$ preserves tails of $\overline{Q_3}$-representations of numbers if for any $x\in[0,1)$ there exist positive integers $k=k(x)$ and $m=m(x)$ such that $a_{k+n}(x)=a_{m+n}(f(x))$ for all $n\in\mathbb{N}$. It is clear that functions preserving tails of $\overline{Q_3}$-representations of numbers form an infinite set.

In the set $Z^{\overline{Q_3}}_{[0;1)}$ we consider the operator $w$ of the left shift of the digits of the representation of number. It correspondence (reflecting) the sequence $(a_1,a_2,\ldots,a_n,\ldots)$ to the sequence $(a_2,a_3,\ldots,a_n,\ldots)$. Due to the correctness of the definition of $\overline{Q_3}$-representation, the operator $w$ generates a function defined by equality $w(\Delta^{\overline{Q_3}}_{a_1a_2\ldots a_n\ldots})=\Delta^{\overline{Q_3}}_{a_2a_3\ldots a_n\ldots}$.

This operator has many invariant numbers: $\Delta^{\overline{Q_3}}_{(i)}$, $i=\{1,2,3,\ldots\}$. It has the property of surjectivity, but is not injective, because the prototype of the point having the representation $\Delta^{\overline{Q_3}}_{a_1a_2\ldots a_n\ldots}$ are points $\Delta^{\overline{Q_3}}_{0a_1a_2\ldots a_n\ldots}$, $\Delta^{\overline{Q_3}}_{1a_1a_2\ldots a_n\ldots}$, $\Delta^{\overline{Q_3}}_{2a_1a_2\ldots a_n\ldots}$,...,$\Delta^{\overline{Q_3}}_{ia_1a_2\ldots a_n\ldots}$.

**Theorem 4.** *The function $w(x)$ has a countable set of points of jump discontinuity at $Q_3$-rational numbers of the half-segment $[0,1)$.*

*Proof.* From Lemma 1 it follows that the geometry of the $\overline{Q_3}$-representation is not self-similar. Consider the effect of the function $w(x)$ on the $Q_3$-representation of points from $[0,1)$. From equality (1) it follows that the values of this function $w(x)$ are obtained from the $Q_3$-representation of the argument by the following change of digits: 0 by 2; 1 by 0; and 2 by 1. Therefore, the function $w(x)$ is discontinuous at $Q_3$ -rational points. This is easily illustrated by the example of an arbitrary $Q_3$-rational number.

Consider the number $x = q_0^2$. It has two formally different $Q_3$-representations, namely $\Delta^{Q_3}_{01(0)}$ and $\Delta^{Q_3}_{00(2)}$. The number $x$ has the unique representation in the $\overline{Q_3}$-representation, namely $\Delta^{\overline{Q_3}}_{110(100)}$. Then $w(x) = w(\Delta^{\overline{Q_3}}_{110(100)}) = \Delta^{\overline{Q_3}}_{10(100)} = \Delta^{Q_3}_{0(2)} = q_0$.

Investigate the behavior of the function $w(x)$ in the left $\varepsilon$-neighborhood of number $x$. If $x \in (q_0^2 - \varepsilon; q_0^2)$, then respectively $x = \Delta^{\overline{Q_3}}_{20n(100)}$, where $n \to \infty$. Then $w(\Delta^{\overline{Q_3}}_{20n(100)}) = \Delta^{\overline{Q_3}}_{0n(100)}$. The condition $x \to q_0^2 - 0$ is equivalent to the condition $n \to \infty$. Hence $\lim\limits_{x \to q_0^2 - 0} w(x) = \Delta^{Q_3}_{111\ldots1\ldots} = \frac{q_0}{1-q_1}$. The inequality $q_0 \neq \frac{q_0}{1-q_1}$ is true for all $q_0$, $q_1$. So number $x = q_0^2$ is a jump point with jump $\frac{q_0 q_1}{1-q_1}$.

The similar assertion is valid for the other $Q_3$-rational numbers.

It is known that the set of $Q_3$-rational numbers from $[0,1)$ is countable. Then the set of point of discontinuity of function $w(x)$ is also countable. Since the range of admissible values belongs to the segment $[0,1]$, the points of discontinuity of the function $w(x)$ are of the first kind. □

**Corollary 1.** *The operator $w(x)$ of the left shift of $\overline{Q_3}$-digits at the points of discontinuity has different jumps.*

*Proof.* As shown above, the number $x = q_0^2$ is the point of jump discontinuity with jump $\frac{q_0 q_1}{1-q_1}$.

Consider a few more $Q_3$-rational numbers. Let $x = q_0 + q_1$. It has two formally different $Q_3$-representations, namely $\Delta^{Q_3}_{1(2)}$ and $\Delta^{Q_3}_{2(0)}$. The number $x$ has the unique $\overline{Q_3}$-representation, namely $\Delta^{\overline{Q_3}}_{001(100)}$. Then $w(x) = w(\Delta^{\overline{Q_3}}_{001(100)}) = \Delta^{\overline{Q_3}}_{01(100)} = \Delta^{Q_3}_{1(2)} = q_0 + q_1$. We will investigate the behavior of the function $w(x)$ in the left $\varepsilon$-neighborhood of the number $x$. If $x \in (q_0 + q_1 - \varepsilon; q_0 + q_1)$, then $x = \Delta^{\overline{Q_3}}_{01n(100)}$, where $n \to \infty$. Then $w(\Delta^{\overline{Q_3}}_{01n(100)}) = \Delta^{\overline{Q_3}}_{1n(100)}$. The condition $x \to q_0 + q_1 - 0$ is equivalent to the condition $n \to \infty$. Therefore $\lim\limits_{x \to q_0 + q_1 - 0} w(x) = \Delta^{Q_3}_{0111\ldots1\ldots} = \frac{q_0^2}{1-q_1}$. The inequality $q_0 + q_1 \neq \frac{q_0^2}{1-q_1}$ is true for all $q_0$, $q_1$. So the number $x = q_0 + q_1$ is the point of jump discontinuity with a jump $q_0 + q_1 - \frac{q_0^2}{1-q_1}$.

Let $x = q_0 + q_1 + q_1 \cdot q_2$. Using similar considertions we can show that this is the point of jump discontinuity with a jump of $q_0^2 + \frac{q_0 \cdot q_1 \cdot q_2}{1-q_1}$. And so on.

Therefore, the function $w(x)$ has different jumps at points of discontinuity. □

**Corollary 2.** *The left shift functions $w^2(x)$, $w^3(x)$, ..., $w^k(x)$ by 2, 3, ..., $k$ $\overline{Q_3}$-digits have a countable set of points of jump discontinuity at $Q_3$-rational numbers of the half-segment $[0,1)$.*

**Corollary 3.** *The left shift functions $w^3(x)$, $w^6(x)$, ..., $w^{3k}(x)$ by 3, 6, ..., $3k$ $\overline{Q_3}$-digits have a countable set of points of jump discontinuity with a jump 1 at $Q_3$-rational numbers of the half-segment $[0,1)$.*

Let the element $i$ of the alphabet $A = \{0, 1, 2, \ldots\}$ be fixed.

In the set $Z^{\overline{Q_3}}_{[0;1)}$ we consider the operator $\delta_i$ of the right shift of the digits in the $\overline{Q_3}$-representation of the number. It maps the sequence $(a_1, a_2, \ldots, a_n, \ldots)$ to the sequence $(i, a_1, a_2, a_3, \ldots, a_n, \ldots)$.

The operator $\delta_i$ is denoted by the equality $\delta_i(\Delta^{\overline{Q_3}}_{a_1a_2\ldots a_n\ldots}) = \Delta^{\overline{Q_3}}_{ia_1a_2a_3\ldots a_n\ldots}$. This defines a countable class of functions $y = \delta_i(x)$, $i \in \mathbb{N}$.

Obviously, this operator has infinite amount of invariant numbers: $\Delta^{\overline{Q_3}}_{(i)}$, $i = \{1, 2, 3, \ldots\}$.

**Theorem 5.** *The function $\delta_i(x)$ has a countable set of points of jump discontinuity at $Q_3$-rational numbers of the half-segment $[0, 1)$.*

*Proof.* From Lemma 1 it follows that the geometry of the $\overline{Q_3}$-representation is not self-similar. From the definitions of $\delta_i(x)$ and $\overline{Q_3}$-representation of real numbers, we have

$$x = \Delta^{\overline{Q_3}}_{a_1a_2\ldots a_n\ldots} = \Delta^{Q_3}_{\underbrace{0\ldots0}_{a_1}\underbrace{1\ldots1}_{a_2}\underbrace{2\ldots2}_{a_3}\ldots\underbrace{0\ldots0}_{a_{3n-2}}\underbrace{1\ldots1}_{a_{3n-1}}\underbrace{2\ldots2}_{a_{3n}}\ldots},$$

where $a_n$ is the "length" of a series of identical consecutive $Q_3$-digits, and

$$\delta_i(x) = \Delta^{\overline{Q_3}}_{ia_1a_2\ldots a_n\ldots} = \Delta^{Q_3}_{\underbrace{0\ldots0}_{i}\underbrace{1\ldots1}_{a_1}\underbrace{2\ldots2}_{a_2}\ldots\underbrace{0\ldots0}_{a_{3n-3}}\underbrace{1\ldots1}_{a_{3n-2}}\underbrace{2\ldots2}_{a_{3n-1}}\ldots}.$$

Next, we repeat arguments which are similar to the proof of Theorem 4. We obtain that $\delta_i(x)$ has a countable set of points of discontinuity of the first kind at $Q_3$-rational numbers. □

**5. Hausdorff–Besicovitch fractal dimension.** One of the traditional problems of the Hausdorff–Besicovitch dimension theory [13] is the problem of whether the class of sets $\Phi$ is sufficient for $\alpha_0(E, \Phi) = \alpha_0(E)$.

Let $W$ be the class of all connected sets, which are unions of cylinders of the same rank, belonging to the same cylinder of the previous rank, i.e., sets of the form:

$$\Delta^{\overline{Q_3}}_{c_1c_2\ldots c_m}; \quad \bigcup_{i=n}^{\infty} \Delta^{\overline{Q_3}}_{c_1c_2\ldots c_m i}; \quad \bigcup_{i=1}^{n} \Delta^{\overline{Q_3}}_{c_1c_2\ldots c_m i}; \quad \bigcup_{i=k}^{n} \Delta^{\overline{Q_3}}_{c_1c_2\ldots c_m i}$$

for all $k, m, n \in \mathbb{N}$ and sets of natural numbers $(c_1, c_2, \ldots, c_m)$.

Denote by $W_\varepsilon$ the class of sets from $W$ whose diameters do not exceed $\varepsilon$.

**Lemma 3.** *For an arbitrary interval $h \subset [0; 1)$, there are no more than four sets from $W_{|h|}$ that cover $h$ and have diameters at most $|\, h \,|$.*

*Proof.* Let $h = (a, b)$. The numbers $a$ and $b$ may belong to different cylinders of the first rank or one cylinder of the first rank.

Let $a \in \Delta^{\overline{Q_3}}_{a_1}$, $b \in \Delta^{\overline{Q_3}}_{b_1}$, $c = \max \Delta^{\overline{Q_3}}_{a_1}$, $d = \min \Delta^{\overline{Q_3}}_{b_1}$. According to the condition $a < b$, we have $a_1 > b_1$ and $a < d < b$. Then $a_1 - b_1 > 1$ or $a_1 - b_1 = 1$.

Let us consider the case $a_1 - b_1 > 1$. Then $(a, b) = (a, d] \cup (d, b)$. If $a = \min \Delta^{\overline{Q_3}}_{a_1}$, then $[a, d] = \bigcup\limits_{i=b_1+1}^{a_1} \Delta^{\overline{Q_3}}_{i} \subset W_{d-a} \subset W_{b-a}$. If $a \in \nabla^{\overline{Q_3}}_{a_1}$, then $(a, d)$ is covered by sets from $W_{d-a}$, namely: $\Delta^{\overline{Q_3}}_{a_1}$ and $\bigcup\limits_{i=b_1+1}^{a_1-1} \Delta^{\overline{Q_3}}_{i}$. For covering of $[a, d]$ it is sufficient to choose sets from $W_{d-a}$, and hence from $W_{b-a}$.

If $b = \max \Delta^{\overline{Q_3}}_{b_1}$, then $[d, b] = \Delta^{\overline{Q_3}}_{b_1} \subset W_{b-d} \subset W_{b-a}$. If $b \in \nabla^{\overline{Q_3}}_{b_1}$, then we consider the cylinders of the second rank $\Delta^{\overline{Q_3}}_{b_1 j}$ which belong to $\Delta^{\overline{Q_3}}_{b_1}$.

If $b = \max \Delta^{\overline{Q_3}}_{b_1 n}$, then $[d,b] = \bigcup\limits_{j=1}^{n} \Delta^{\overline{Q_3}}_{b_1 j} \subset W_{b-d}$. If $b \in \nabla^{\overline{Q_3}}_{b_1 n}$, then $[d,b]$ cover:

1) $\bigcup\limits_{j=1}^{n-1} \Delta^{\overline{Q_3}}_{b_1 j}$ and $\Delta^{\overline{Q_3}}_{b_1 n}$ if $n > 1$; 2) $\Delta^{\overline{Q_3}}_{b_1 1}$ if $n = 1$.

Therefore, two sets from $W_{b-a}$ are sufficient to cover $[d,b]$. At most four sets from $W_{b-a}$ are enough to cover $[a,b]$.

Let us consider the case $a_1 - b_1 = 1$. Then the subcases $a_1 = 1$, $b_1 = 0$ or $a_1 > 1$, $a_1 = b_1 + 1$ are possible.

Let $a_1 = 1$, $b_1 = 0$, then $c = d$, $a \in \nabla^{\overline{Q_3}}_{a_1}$.

Consider $[a,d]$. If $a = \min \Delta^{\overline{Q_3}}_{a_1 k}$, then $[a,d] = \bigcup\limits_{j=k}^{\infty} \Delta^{\overline{Q_3}}_{a_1 j} \subset W_{d-a} \subset W_{b-a}$. If $a \in \nabla^{\overline{Q_3}}_{a_1 k}$, then $[a,d]$ is covered by two sets from $W_{b-a}$: $\Delta^{\overline{Q_3}}_{a_1 k}$ and $\bigcup\limits_{j=k+1}^{\infty} \Delta^{\overline{Q_3}}_{a_1 j}$ Therefore, two sets are enough to cover $[a,d]$.

Consider $[d,b]$. If $b = \max \Delta^{\overline{Q_3}}_{b_1 n}$, then $[d,b] = \bigcup\limits_{j=1}^{n} \Delta^{\overline{Q_3}}_{b_1 j} \subset W_{b-d}$. If $b \in \nabla^{\overline{Q_3}}_{b_1 n}$ and $n > 1$, then $[d,b]$ is covered by two sets from $W_{b-a}$: $\Delta^{\overline{Q_3}}_{b_1 n}$ and $\bigcup\limits_{j=1}^{n-1} \Delta^{\overline{Q_3}}_{b_1 j}$, because $|\Delta^{\overline{Q_3}}_{b_1 n}| < |\Delta^{\overline{Q_3}}_{b_1 1}|$, $\Delta^{\overline{Q_3}}_{b_1 1} \subset [d,b]$. If $b \in \nabla^{\overline{Q_3}}_{b_1 1}$, then consider the cylinders $\Delta^{\overline{Q_3}}_{b_1 1 j}$ of the third rank, which belong to $\Delta^{\overline{Q_3}}_{b_1 1}$. In this case, $[d,b]$ is covered by two sets from $W_{b-a}$, namely:

1) if $b = \max \Delta^{\overline{Q_3}}_{b_1 1 s}$, then $\bigcup\limits_{j=s}^{\infty} \Delta^{\overline{Q_3}}_{b_1 1 j}$; 2) if $b \in \nabla^{\overline{Q_3}}_{b_1 1 s}$, then $\bigcup\limits_{j=s+1}^{\infty} \Delta^{\overline{Q_3}}_{b_1 1 j}$ and $\Delta^{\overline{Q_3}}_{b_1 1 s}$.

Therefore, two sets from $W_{b-a}$ are enough to cover $[d,b]$. Four sets from $W_{b-a}$ are enough to cover $[a,b]$.

If $a_1 > 1$ and $a_1 = b_1 + 1$, then the proof of this case is similar to the proof of the case $a_1 - b_1 > 1$. Therefore, two sets from $W_{b-a}$ are enough to cover $[d,b]$. Four sets from $W_{b-a}$ are enough to cover $[a,b]$.

If $a$ and $b$ belong to one cylinder of the first rank, then there is a cylinder $\Delta^{\overline{Q_3}}_{c_1 c_2 \ldots c_m}$ of rank $m$, which belongs to the numbers $a$ and $b$, but there is not cylinder of rank $m+1$, which would belong to $a$ and $b$. Next, to prove the lemma, it is necessary to repeat the above considerations, where the role $[0,1)$ will be played by the cylinder $\Delta^{\overline{Q_3}}_{c_1 c_2 \ldots c_m}$ of rank $m$.

Lemma 3 is proved. □

**Theorem 6.** *Class of sets $W$ is sufficient to determine the Hausdorff–Besicovitch dimension of an arbitrary Borel set $E \subset [0,1)$, i.e., $\alpha_0(E,W) = \alpha_0(E)$.*

*Proof.* From Lemma 3 it follows $m^{\alpha}_{\varepsilon}(E,W) \leq 4 m^{\alpha}_{\varepsilon}(E)$. In fact, for an arbitrary segment $h$ from the covering $E$, there exist at most four sets $w_1$, $w_2$, $w_3$, $w_4$ from $W$. For these sets $|w_i|^{\alpha} \leq |h|^{\alpha}$ for an arbitrary $\alpha \in (0,1)$.

On the other hand, $m^{\alpha}_{\varepsilon}(E) \leq m^{\alpha}_{\varepsilon}(E,W)$. Since in the definition of $m^{\alpha}_{\varepsilon}(E)$ the infinum is taken for a wider class of coverings, which includes sets from $W$. Then, $m^{\alpha}_{\varepsilon}(E) \leq m^{\alpha}_{\varepsilon}(E,W) \leq 4 m^{\alpha}_{\varepsilon}(E)$ for any $\varepsilon > 0$. The transition to the limits gives $H^{\alpha}(E) \leq H^{\alpha}(E,W) \leq 4 H^{\alpha}(E)$, i.e., $H^{\alpha}(E)$ and $H^{\alpha}(E,W)$ on $\alpha$ acquire the values 0 and $\infty$ simultaneously. This means that there is an equality $\alpha_0(E,W) = \alpha_0(E)$.

Theorem 6 is proved. □

**6. Some problems of probabilistic number theory.** Probabilistic theory of representations of real numbers solves probabilistic problems using different systems for representing of numbers. Its important component is the study of probability distributions on sets of real numbers determined by the conditions for their representation in a system. In particular, we will study random variables, for which the digits of the representation are random variables with predefined distributions.

**Theorem 7.** *If the digits of the $\xi_k$ $\overline{Q_3}$-representation of the random variable $\xi = \Delta^{\overline{Q_3}}_{\xi_1\xi_2\ldots\xi_n\ldots}$ are independent random variables that acquire values of $1, 2, ..., i, ...$, accordingly with probabilities $p_{1k}, p_{2k}, ..., p_{ik}, ...$ ($\sum\limits_{i=1}^{\infty} p_{ik} = 1$, $k \in \mathbb{N}$), then the distribution $\xi$ is either purely discrete or purely continuous, and purely discrete if and only if $M = \prod\limits_{k=1}^{\infty} \max\limits_i \{p_{ik}\} > 0$.*

*The point spectrum of a discretely distributed random variable $\xi$ consists of a point $x_0$ such that $p_{a_j(x_0)j} = \max\limits_i \{p_{ik}\}$ for all numbers $x$ that have the property $p_{a_j(x_0)j} > 0$ for any $j \in \mathbb{N}$ and there exists such $m \in \mathbb{N}$, that $a_j(x) = a_j(x_0)$ for $j \geq m$.*

*Proof.* The random variable $\xi_k$ does not acquire values from the set of numbers that have finite $\overline{Q_3}$-representations. Further in our reasoning, we neglect the numbers of this set. The remaining numbers from $[0;1)$ have the unique $\overline{Q_3}$-representation. From the independence of $\xi_k$ and the uniqueness of the $\overline{Q_3}$-representation it follows that $P\{\xi = \Delta^{\overline{Q_3}}_{c_1c_2\ldots c_m\ldots}\} = \prod\limits_{k=1}^{\infty} p_{c_k k}$, i.e., $P\{\xi = x\} = \prod\limits_{j=1}^{\infty} p_{a_j(x)j}$.

If $M > 0$, then the distribution $\xi$ is purely discrete. Since $P\{\xi = x_0\} = M$, then $P\{\xi = x_0\} > 0$.

If $p_{a_k(x')k} > 0$ for any $k \in \mathbb{N}$ and the $\overline{Q_3}$-representation of the number $x'$ differs from the representation of the $x_0$ on at most the first $(m-1)$ $\overline{Q_3}$-symbols, then

$$P\{\xi = x'\} = \prod_{k=1}^{m-1} p_{a_k(x')k} \cdot \prod_{k=m}^{\infty} p_{a_k(x_0)k} = \prod_{k=1}^{m-1} p_{a_k(x')k} \cdot \frac{M}{\prod\limits_{k=1}^{m} p_{a_k(x_0)k}}.$$

Let $A_m$ be the set of all numbers $x'$, where $\overline{Q_3}$-digits of $x'$ coincide with $\overline{Q_3}$-digits of the number $x_0$, starting with $m$. Then the sequence of sets $A_m$ has the properties:

$$\{x_0\} = A_1 \subset A_2 \subset \ldots \subset A_m \subset A_{m+1} \subset \ldots,$$

$$P\{\xi \in A_m\} = \sum_{a_1 \in \mathbb{N}} \ldots \sum_{a_{m-1} \in \mathbb{N}} \left( \prod_{k=1}^{m-1} p_{a_k(x')k} \cdot \frac{M}{\prod\limits_{k=1}^{m} p_{a_k(x_0)k}} \right) = \frac{M}{\prod\limits_{k=1}^{m} p_{a_k(x_0)k}} \to 1 \ (m \to \infty).$$

Therefore, the countable set $A = \lim\limits_{m\to\infty} A_m = \bigcup\limits_{m=1}^{\infty} A_m$ is the support of the distribution of random variables $\xi$, i.e., the distribution is discrete.

If $\xi$ has a discrete distribution, then there exists $x'$ such that

$$0 < P\{\xi = x'\} = \prod_{k=1}^{\infty} p_{a_k(x')k} \leq \prod_{k=1}^{\infty} \max_i \{p_{ik}\} = M,$$

i.e., $M > 0$. Theorem 7 is proved. □

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