J. G. Anaya, E. Castañeda-Alvarado, J. A. Martínez-Cortez

INDUCED MAPPINGS ON $C_n(X)/C_{nK}(X)$

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Given a continuum X and $n \in \mathbb{N}$. Let $C_n(X)$ be the hyperspace of all nonempty closed subsets of X with at most n components. Let $C_{nK}(X)$ be the hyperspace of all elements in $C_n(X)$ containing K where K is a compact subset of X. The quotient space $C_n(X)/C_{nK}(X)$ will be denote by $C^n_K(X)$. Given a mapping $f\colon X \to Y$ between continua, let $C_n(f)\colon C_n(X) \to C_n(Y)$ be the mapping induced by f, defined by $C_n(f)(A) = f(A)$. We denote the natural induced mapping between $C^n_K(X)$ and $C^n_{f(K)}(Y)$ by $C^n_K(f)$. In this paper, we study relationships among the mappings f, $C_n(f)$ and $C^n_K(f)$ for the following classes of mappings: almost monotone, atriodic, confluent, joining, light, monotone, open, OM, pseudo-confluent, quasi-monotone, semi-confluent, strongly freely decomposable, weakly confluent, and weakly monotone.

1. Introduction. A *continuum* is a nonempty compact connected metric space. A *subcontinuum* of a continuum X is a subset of X which is a continuum. A *mapping* is a continuous function. We will denote by $\mathbb{N}$ the set of positive integers, by I the unit interval $[0,1]$, and by S^1 the unit circle $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.

Given a continuum X and $n \in \mathbb{N}$, we consider the following *hyperspaces* of X

$$2^X = \{A \subset X : A \text{ is nonempty and closed in } X\},$$
$$C_n(X) = \{A \in 2^X : A \text{ has at most } n \text{ components}\},$$
$$F_n(X) = \{A \in 2^X : A \text{ has at most } n \text{ points }\}.$$

All the hyperspaces topologized with the Hausdorff metric (see the definition below). Given a nonempty compact subset K of X, the subspace $C_{nK}(X)$ of $C_n(X)$ defined by

$$C_{nK}(X) = \{A \in C_n(X) \ : \ K \subset A\}$$

is called the *containment hyperspace for K in $C_n(X)$*.

The hyperspace $C_n(X)$ is called the *n-fold hyperspace* of X, his structure topologic is different to other hyperspaces, see [22] and [23]. For example, by [18, Lemma 2.3, p. 349], $C_2(I)$ is not homeomorphic to $C_2(S^1)$. In fact, $C_2(I)$ is homeomorphic to a 4-dimensional cell (see [18, Lemma 2.2, p. 349]) and $C_2(S^1)$ is homeomorphic to the cone over the solid torus (see [19]). The hyperspace $C_1(X)$ is called the *hyperspace of subcontinua*, some geometric models of $C_1(X)$ are (see [20, Chapter II]):

- $C_1(I)$ is a triangle;
- $C_1(S^1)$ is the unit disk;

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- $C_1(T)$ is a cube with three triangles, where T is the cone over three points;
- $C_1(X)$ is the n-dimensional polyhedron built by attaching n two dimensional cell with an n-dimensional cell where X is the cone over n points, $n \in \mathbb{N}$.
- $C_1(P)$ is a 3-dimensional polyhedron (see [20, Figure 6, p. 37]), where P is the union of a simple closed curve and an arc whose intersection is one of the end points of the arc.

For a continuum X, since $C_{nK}(X)$ is a nonempty closed subset of $C_n(X)$,

$$\{C_{nK}(X)\} \cup \{\{A\} : A \in C_n(X) - C_{nK}(X)\}$$

is an upper semi-continuous decomposition of $C_n(X)$. By [29, Theorem 3.10, p. 40], the space $C_n(X)/C_{nK}(X)$ is a continuum, which is denoted by $C_K^n(X)$, where π_K^X stands for the quotient mapping $\pi_K^X\colon C_n(X) \to C_K^n(X)$. For each $A \in C_n(X) - C_{nK}(X)$ we denote the class of A by $\mathcal{A}$, and let $C_{nK}^X = \pi_K^X(C_{nK}(X))$. Thus, π_K^X is given by

$$\pi_K^X(A) = \begin{cases} \mathcal{A} & \text{if } A \notin C_{nK}(X), \\ C_{nK}^X & \text{if } A \in C_{nK}(X). \end{cases}$$

In 1979 S. B. Nadler Jr., see [28], began the study of the quotient space $C_1(X)/F_1(X)$, which he called the *hyperspace suspension* of X. Later, in 2004, R. Escobedo, M. de J. López and S. Macías extended the study of hyperspace suspension in [14].

Subsequently, S. Macías generalized the study of hyperspace suspension, considering the quotient space $C_n(X)/F_n(X)$, which he called the *n-fold hyperspace suspension* of X, see [24], continuing with the study in 2006, see [25]. In the year 2008, J. C. Macías analyzes the quotient space $C_n(X)/F_1(X)$, which he called the *n-fold pseudo-hyperspace suspension* of X, see [21]. J. Camargo and S. Macías in 2016 considered the quotient space $C_n(X)/C_1(X)$, they show several of their properties, see [9]. With respect to the space $C_K^n(X)$ in [2] is demonstrated that $C_K^n(I)$ is homeomorphic to the suspension over $C_{nK}(I)$, where $K \in \{\{0\}, \{1\}\}$. In particular, $C_K^n(I)$ is homeomorphic to a 2-dimensional cell for $n = 1$ (see [2, Corollary 3.11]). Other example is that $C_K^n(S^1)$ is homeomorphic to a 2-dimensional cell for $n = 1$ and $K \in 2^{S^1}$ (see [2, Theorem 3.13]).

On the other hand, given a mapping $f\colon X \to Y$ between continua, the mapping

$$C_n(f)\colon C_n(X) \to C_n(Y)$$

defined by $C_n(f)(A) = f(A)$ for each $A \in C_n(X)$ is called the *induced mapping by f*. Let $C_K^n(f)\colon C_K^n(X) \to C_K^n(Y)$ be the function defined by

$$C_K^n(f)(\pi_K^X(A)) = \pi_{f(K)}^Y(C_n(f)(A)) = \pi_{f(K)}^Y(f(A))$$

for each $A \in C_n(X)$. By [13, Theorem 4.3, p. 126], $C_K^n(f)$ is a mapping.

Let $\mathbb{A}$ be a class of mappings between continua. A general problem is to determine all possible relationships among the following statements:
(1) $f \in \mathbb{A}$; (2) $C_n(f) \in \mathbb{A}$; (3) $C_K^n(f) \in \mathbb{A}$ for each $K \in 2^X$;
(4) $C_K^n(f) \in \mathbb{A}$ for some $K \in 2^X$.

There are particular results concerning this problem, which relate (1) and (2). Readers especially interested in this topic are referred, for example, to [5], [7], [8], [11], [12], [16], [17]. Regarding induced mappings in quotient hyperspaces we refer the reader, for example, to [1], [3], [4], [6], [10].

Following this line of research, in this paper we study interrelations among the statements (1)–(4), for the following classes of mappings: almost monotone, atriodic, confluent, joining, light, monotone, open, OM, pseudo-confluent, quasi-monotone, semi-confluent, strongly freely decomposable, weakly confluent, and weakly monotone.

2. Definitions and notations. Given a topological space Z, we denote the closure and interior of a subset A of Z by $\mathrm{Cl}_Z(A)$ and $\mathrm{Int}_Z(A)$, respectively. Let X be a continuum, with metric d, and $\epsilon > 0$. The open ball in X of radius ϵ and center x will be denoted by $B^d_\epsilon(x)$. The hyperspace 2^X is considered with the Hausdorff metric induced by d, which is denoted by H_d and defined as follows (see [27, (0.1), p. 1] or [20, Definition 2.1, p. 11]): for any $A, B \in 2^X$,

$$H_d(A,B) = \inf\{\epsilon > 0\colon A \subset N_d(B,\epsilon) \text{ and } B \subset N_d(A,\epsilon)\}, \text{ where } N_d(A,\epsilon) = \bigcup_{x\in A} B^d_\epsilon(x).$$

Given a mapping $f\colon X \to Y$ between continua. The *induced function from* 2^X into 2^Y is the function f^* defined by $f^*(A) = f(A)$ for each $A \in 2^X$. For each $\mathcal{H}(X) \in \{2^X, C_n(X), F_n(X)\}$, the *induced function from* $\mathcal{H}(X)$ into $\mathcal{H}(Y)$ is the function $\mathcal{H}(f) = f^*|_{\mathcal{H}(X)}\colon \mathcal{H}(X) \to \mathcal{H}(Y)$ which is a mapping (see [20, Theorem 13.3, p. 106]).

Let A, $B \in 2^X$. An *order arc* from A to B is a mapping $\alpha\colon I \to 2^X$ such that $\alpha(0) = A$, $\alpha(1) = B$, and $\alpha(r)$ is a proper subset of $\alpha(s)$ whenever $r < s$ (see [27, (1.2)-(1.8), p. 57-59]). For any finitely many subsets $U_1, \ldots, U_r$ of X, we define

$$\langle U_1,\ldots,U_r\rangle = \left\{A \in 2^X\colon\ A \subset \bigcup_{i=1}^r U_i,\ A\cap U_i \neq \varnothing, \text{ for each } i = 1,\ldots,r\right\}.$$

The set $\{\langle U_1,\ldots,U_r\rangle\colon$ for each $i \in \{1,\ldots,r\}$, U_i is an open subset of X, $r \in \mathbb{N}\}$ is a base for a topology on 2^X. This topology is called the *Vietoris topology* and matches with the topology induced by H_d (see [20, Theorem 3.2, p. 18]). In this paper, $\langle U_1,\ldots,U_r\rangle_n$ denote the set $\langle U_1,\ldots,U_r\rangle \cap C_n(X)$.

An onto mapping $f\colon X \to Y$ between continua is said to be:

- *almost monotone* provided that for each subcontinuum Q of Y with $\mathrm{Int}_Y(Q) \neq \varnothing$, $f^{-1}(Q)$ is connected;
- *atriodic* if for every subcontinuum Q of Y, there exist two components C and D of $f^{-1}(Q)$ such that $f(C)\cup f(D) = Q$ and for each component E of $f^{-1}(Q)$, we have that either $f(E) = Q$, or $f(E) \subset f(C)$ or $f(E) \subset f(D)$;
- *confluent* if for every subcontinuum K of Y and for each component M of $f^{-1}(K)$, $f(M) = K$;
- *freely decomposable* if whenever A and B are proper subcontinua of Y such that $Y = A\cup B$, then there exist two proper subcontinua A' and B' of X, such that $X = A'\cup B'$, $f(A') \subset A$ and $f(B') \subset B$;
- *joining* provided that for each subcontinuum Q of Y and for any two components C and D of $f^{-1}(Q)$, we have that $f(C)\cap f(D) \neq \varnothing$;
- *light* if $f^{-1}(y)$ is totally disconnected for each $y \in Y$;
- *monotone* if $f^{-1}(y)$ is connected for each $y \in Y$;
- *open* if $f(U)$ is open in Y for each open subset U of X;
- *OM* if there exist a continuum Z and mappings $g\colon X \to Z$ and $h\colon Z \to Y$ such that $f = h\circ g$, g is monotone and h is open;
- *pseudo-confluent* provided that for each irreducible subcontinuum B of Y, there exists a component C of $f^{-1}(B)$ such that $f(C) = B$;

- *quasi-monotone* provided that for any subcontinuum B of Y with $\mathrm{Int}_Y(B) \neq \varnothing$, $f^{-1}(B)$ has only finitely many components and each of these components maps onto B under f;
- *semi-confluent* if for every subcontinuum B of Y and every pair of components C and D of $f^{-1}(B)$, either $f(C) \subset f(D)$ or $f(D) \subset f(C)$;
- *semi-open* if for every open subset U of X, $\mathrm{Int}_Y(f(U)) \neq \varnothing$;
- *strongly freely decomposable* if whenever A and B are proper subcontinua of Y such that $Y = A \cup B$, we obtain that $f^{-1}(A)$ and $f^{-1}(B)$ are connected;
- *weakly confluent* if for each subcontinuum K of Y, there exists a subcontinuum M of X such that $f(M) = K$;
- *weakly monotone* provided that for each subcontinuum B of Y with $\mathrm{Int}_Y(B) \neq \varnothing$, each component of $f^{-1}(B)$ is mapped by f onto B.

3. Preliminary results. Let X be a continuum and let L be a subcontinuum of X. We denote by X/L the quotient space obtained by shrinking L to a point. By [29, Theorem 3.10, p. 40], X/L is a continuum. Let X, Y be continua, let L be a subcontinuum of X, and let $f\colon X \to Y$ be an onto mapping. Let $q_X\colon X \to X/L$ and $q_Y\colon Y \to Y/f(L)$ be the quotient mappings. We will denote $q_X(L)$ and $q_Y(f(L))$ by L_X and L_Y, respectively. Note that f induces a function $\tilde{f}\colon X/L \to Y/f(L)$ (see [13, Theorem 7.7, p. 17]) given by

$$\tilde{f}(\mathcal{A}) = \begin{cases} q_Y(f((q_X)^{-1}(\mathcal{A}))) & \text{if } \mathcal{A} \neq L_X, \\ L_Y & \text{if } \mathcal{A} = L_X. \end{cases}$$

The continuity of $\tilde{f}$ follows from [13, Theorem 4.3, p. 126]. Observe that $\tilde{f} \circ q_X = q_Y \circ f$.

Suppose that $\mathbb{A}$ is any of the following classes of mappings between continua: monotone, OM, confluent, semi-confluent, weakly confluent, pseudo-confluent, quasi-monotone, weakly monotone, joining, almost monotone, atriodic, freely decomposable or strongly freely decomposable. With the previous notation, we have the following result.

Proposition 1. *If $f \in \mathbb{A}$, then $\tilde{f} \in \mathbb{A}$.*

Proof. In [4, Theorem 3.2, p. 493] is proved that if f is either almost monotone, or atriodic, or freely decomposable or strongly freely decomposable, then $\tilde{f}$ is almost monotone, or atriodic, or freely decomposable or strongly freely decomposable, respectively. Let $\mathbb{A}$ be one of the other classes of mappings of the statement. Since q_Y is monotone, $q_Y \in \mathbb{A}$. By [26, (5.1), (5.4), (5.5), (5.6)], and Propositions 4.1, 4.3 and 4.4 of [6], $q_Y \circ f \in \mathbb{A}$. Now, by [26, (5.15), (5.16), (5.19), (5.20) and (5.21)], $\mathbb{A}$ has the composition factor property. Since $q_Y \circ f = \tilde{f} \circ q_X$, $\tilde{f} \circ q_X \in \mathbb{A}$. Therefore $\tilde{f} \in \mathbb{A}$. □

Since $q_X|_{X-L}$ and $q_Y|_{Y-f(L)}$ are homeomorphisms and $f|_{f^{-1}(Y-f(L))} = q_Y^{-1}|_{Y-f(L)} \circ \tilde{f} \circ q_X$, we have the following proposition.

Proposition 2. *Let $f\colon X \to Y$ be a mapping between continua and let L be a subcontinuum of X.*

(1) If $\tilde{f}$ is confluent, then for each subcontinuum $B \subset Y - f(L)$ and each component A of $f^{-1}(B)$, $f(A) = B$.

(2) If $\tilde{f}$ is weakly confluent, then for each subcontinuum $B \subset Y - f(L)$, there exists A a subcontinuum of X such that $f(A) = B$.

(3) *If $\tilde{f}$ is quasi-monotone (weakly monotone), then for each subcontinuum $B \subset Y - f(L)$ with $\mathrm{Int}_Y(B) \neq \varnothing$ and each component A of $f^{-1}(B)$, $f(A) = B$.*

The following proposition is a consequence of [4, Theorem 3.1, p. 492].

Proposition 3. *Let X, Y be continua and let K be a compact subset of X. If $f\colon X \to Y$ is an onto mapping, then the following hold:*

(1) *The mappings π^X_K and $\pi^Y_{f(K)}$ are monotone;*

(2) *The mappings $\pi^X_K|_{C_n(X)-C_{nK}(X)}\colon C_n(X) - C_{nK}(X) \to C^n_K(X) - \{C^X_{nK}\}$ and $\pi^Y_{f(K)}|_{C_n(Y)-C_{nf(K)}(Y)}\colon C_n(Y)-C_{nf(K)}(Y) \to C^n_{f(K)}(Y)-\{C^Y_{nf(K)}\}$ are homeomorphisms;*

(3) *If $C_{nK}(X)$ and $C_{nf(K)}(Y)$ are nowhere dense in $C_n(X)$ and $C_n(Y)$, respectively, then π^X_K and $\pi^Y_{f(K)}$ are semi-open mappings.*

Lemma 1. *Let $f\colon X \to Y$ be an onto mapping between continua and $n, r \in \mathbb{N}$ such that $r \leq n$. Let $L_1, \ldots, L_r$ be nonempty disjoint closed subsets of Y. For each $i \in \{1, \ldots, r\}$, let M_i be a component of $f^{-1}(L_i)$. Then:*

(1) *$\langle M_1, \ldots, M_r\rangle_n$ is a component of $C_n(f)^{-1}(\langle L_1, \ldots, L_r\rangle_n)$.*

(2) *If M is a component of $f^{-1}(L_i)$ such that $M \neq M_i$ and $r < n$, then $\langle M_1, \ldots, M_r, M\rangle_n$ is a component of $C_n(f)^{-1}(\langle L_1, \ldots, L_r\rangle_n)$.*

(3) *If $K \in 2^X$ and $f(K) \not\subset \bigcup_{i=1}^r L_i$, then $\pi^X_K(\langle M_1, \ldots, M_r\rangle_n)$ is a component of $C^n_K(f)^{-1}(\pi^Y_{f(K)}(\langle L_1, \ldots, L_r\rangle_n))$.*

Proof. The statements (1) and (2) are proved in [1, Proposition 2.4, p. 478]. We prove (3), let $\mathfrak{D}$ be the component of $C^n_K(f)^{-1}(\pi^Y_{f(K)}(\langle L_1, \ldots, L_r\rangle_n))$ containing $\pi^X_K(\langle M_1, \ldots, M_r\rangle_n)$. Note that $\langle M_1, \ldots, M_r\rangle_n \subset (\pi^X_K)^{-1}(\mathfrak{D})$. Since $f(K) \not\subset \bigcup_{i=1}^r L_i$, $\langle L_1, \ldots, L_r\rangle_n \cap C_{nf(K)}(Y) = \varnothing$. Thus, $\pi^Y_{f(K)}(C_{nf(K)}(Y)) \notin \pi^Y_{f(K)}(\langle L_1, \ldots, L_r\rangle_n)$. Hence, $\pi^X_K(C_{nK}(X)) \notin \mathfrak{D}$ and $C_{nK}(X) \cap (\pi^X_K)^{-1}(\mathfrak{D}) = \varnothing$. Since $C^n_K(f) \circ \pi^X_K = \pi^Y_{f(K)} \circ C_n(f)$, $(\pi^X_K)^{-1}(\mathfrak{D}) \subset C_n(f)^{-1}(\langle L_1, \ldots, L_r\rangle_n)$. By (1) of this proposition and (1) of Proposition 3, we have that $(\pi^X_K)^{-1}(\mathfrak{D}) \subset \langle M_1, \ldots, M_r\rangle_n$. Therefore $\mathfrak{D} = \pi^X_K(\langle M_1, \ldots, M_r\rangle_n)$. □

The following result is a consequence of $C_{nK}(X) = \{X\}$, when $K = X$.

Proposition 4. *Let $\mathcal{H}$ be a nondegenerate connected subset of $C_n(X)$. If $X \in \mathcal{H}$, then there exists $K \in 2^X$ such that $C_{nK}(X) \subset \mathcal{H}$.*

Lemma 2. *Let $f\colon X \to Y$ be an onto mapping between continua, $n \in \mathbb{N}$, and let $\mathcal{Q}$ be a closed subset of $C_n(Y)$.*
(1) If $X \notin C_n(f)^{-1}(\mathcal{Q})$, then there exists $m \in \mathbb{N}$ such that $C_{nK}(X) \cap C_n(f)^{-1}(\mathcal{Q}) = \varnothing$ for each $K \in B^H_{\frac{1}{m}}(X)$.
(2) If $X \in C_n(f)^{-1}(\mathcal{Q})$, then there exists $K \in 2^X$ such that $C_{nK}(X) \subset C_n(f)^{-1}(\mathcal{Q})$.

Proof. Suppose that for each $m \in \mathbb{N}$, there exists $K_m \in B^H_{\frac{1}{m}}(X)$ such that $C_{nK_m}(X) \cap C_n(f)^{-1}(\mathcal{Q}) \neq \varnothing$. Then, we may assume that $\{K_m\}_{m\in\mathbb{N}}$ is a sequence in $C_n(X)$ such that $\{K_m\}_{m\in\mathbb{N}}$ converges to X. We consider $L_m \in C_{nK_m}(X) \cap C_n(f)^{-1}(\mathcal{Q})$ for each $m \in \mathbb{N}$. Note that $\{L_m\}_{m\in\mathbb{N}}$ is a sequence in $C_n(f)^{-1}(\mathcal{Q})$ such that $K_m \subset L_m$. Thus, $\{L_m\}_{m\in\mathbb{N}}$ converges to X. Then, $X \in C_n(f)^{-1}(\mathcal{Q})$, this is a contradiction.

To prove (2), let $\mathcal{H}$ be a component of $C_n(f)^{-1}(\mathcal{Q})$ such that $X \in \mathcal{H}$. If $\mathcal{H}$ is degenerate, is easy to verify (2). In another case, by Proposition 4, we conclude (2). □

Lemma 3. *Let $f\colon X \to Y$ be a mapping between continua, $K \in 2^X$ and $n \in \mathbb{N}$. If $f(K) \in F_1(Y)$, then*

$$C_K^n(f)^{-1}(C_{nf(K)}^Y) = \pi_K^X\Big(\bigcup_{p\in f^{-1}(f(K))} C_{n\{p\}}(X)\Big).$$

Proof. Let $\mathcal{A} \in \pi_K^X\Big(\bigcup_{p\in f^{-1}(f(K))} C_{n\{p\}}(X)\Big)$, there exist $p \in f^{-1}(f(K))$ and $B \in C_{n\{p\}}(X)$ such that $\pi_K^X(B) = \mathcal{A}$. Then $C_K^n(f)(\mathcal{A}) = C_K^n(f)(\pi_K^X(B)) = \pi_{f(K)}^Y(C_n(f)(B))$ and $f(p) \in f(B)$. Since $f(K) \in F_1(Y)$, $f(\{p\}) = f(K)$. Thus, $\pi_{f(K)}^Y(C_n(f)(B)) = C_{nf(K)}^Y$. Therefore, $\mathcal{A} \in C_K^n(f)^{-1}(C_{nf(K)}^Y)$.

Now, let $\mathcal{A} \in C_K^n(f)^{-1}(C_{nf(K)}^Y)$. Then $C_K^n(f)(\mathcal{A}) = C_{nf(K)}^Y$. Let $A \in C_n(X)$ such that $\pi_K^X(A) = \mathcal{A}$. Since $C_{nf(K)}^Y = C_K^n(f)(\mathcal{A}) = C_K^n(f)(\pi_K^X(A)) = \pi_{f(K)}^Y(C_n(f)(A)))$ and $f(A) = C_n(f)(A)$, $f(K) \subset f(A)$. Take $p \in f^{-1}(f(K)) \cap A$, thus $A \in C_{n\{p\}}(X)$. Hence, $\mathcal{A} \in \pi_K^X\Big(\bigcup_{p\in f^{-1}(f(K))} C_{n\{p\}}(X)\Big)$. □

Proposition 5. *Let $f\colon X \to Y$ be a mapping between continua, $K \in 2^X$ and $n \in \mathbb{N}$. Then $C_K^n(f)^{-1}(C_{nf(K)}^Y)$ is connected.*

Proof. Suppose that $\mathcal{H}$ and $\mathcal{L}$ are different components of $C_K^n(f)^{-1}(C_{nf(K)}^Y)$. We may assume that $C_{nK}^X \in \mathcal{H}$. By (1) of Proposition 3, $(\pi_K^X)^{-1}(\mathcal{H})$ and $(\pi_K^X)^{-1}(\mathcal{L})$ are disjoint connected subsets of $C_n(X)$ such that $C_{nK}(X) \subset (\pi_K^X)^{-1}(\mathcal{H})$. Now, let $L \in (\pi_K^X)^{-1}(\mathcal{L})$. Note that $C_K^n(f)(\pi_K^X(L)) = C_{nf(K)}^Y$, and for each order arc $\alpha\colon I \to C_n(X)$ from L to X, we have $C_K^n(f)(\pi_K^X(\alpha(I))) = \{C_{nf(K)}^Y\}$. Then, $X \in (\pi_K^X)^{-1}(\mathcal{H}) \cap (\pi_K^X)^{-1}(\mathcal{L})$, this is a contradiction. Therefore, $C_K^n(f)^{-1}(C_{nf(K)}^Y)$ is connected. □

4. Homeomorphism and open mappings.

Theorem 1. *Let $f\colon X \to Y$ be a mapping between continua and $n \in \mathbb{N}$. Then the following conditions are equivalent:*
(1) f is one to one; (2) $C_n(f)$ is one to one; (3) $C_K^n(f)$ is one to one for each $K \in 2^X$;
(4) $C_K^n(f)$ is one to one for some $K \in 2^X$.

Proof. It is easy to see that (1) and (2) are equivalent, (2) implies (3), and (3) implies (4). In order to prove that (4) implies (1), let $x, y \in X$ such that $f(x) = f(y)$. Then $\pi_{f(K)}^Y(\{f(x)\}) = \pi_{f(K)}^Y(\{f(y)\})$. Since $C_K^n(f)(\pi_K^X(A)) = \pi_{f(K)}^Y(f(A))$ for each $A \in C_n(X)$ and $C_K^n(f)$ is one to one, $\pi_K^X(\{x\}) = \pi_K^X(\{y\})$. Then, $\{x\} = \{y\}$ or $K \subset \{x\} \cap \{y\}$. In any case, $x = y$. Therefore f is one to one. □

Theorem 2. *Let $f\colon X \to Y$ be a mapping between continua and $n \in \mathbb{N}$. We consider the following conditions:*
(1)] f is onto; (2) $C_n(f)$ is onto; (3) $C_K^n(f)$ is onto for each $K \in 2^X$;
(4) $C_K^n(f)$ is onto for some $K \in 2^X$.
Then, (2) $\Leftrightarrow$ (3), (3) $\Rightarrow$ (4), (2) $\Rightarrow$ (1), (3) $\Rightarrow$ (1), and (4) $\Rightarrow$ (1).

Proof. Note that (2) implies (3) and (3) implies (4). We will prove that (3) implies (2). Let $B \in C_n(Y)$. If $f^{-1}(B) = X$, then $C_n(f)(X) = B$. Now suppose that $f^{-1}(B) \subsetneq X$, let $K \in 2^X$ such that $K \cap f^{-1}(B) = \varnothing$. Since $C_K^n(f)$ is onto, there exists $\mathcal{A} \in C_K^n(X)$ such that $C_K^n(f)(\mathcal{A}) = \pi_{f(K)}^Y(B)$. Also, there exists $A \in C_n(X)$ such that $\pi_K^X(A) = \mathcal{A}$.

Then, $C^n_K(f)(\mathcal{A}) = C^n_K(f)(\pi^X_K(A)) = \pi^Y_{f(K)}(C_n(f)(A)) = \pi^Y_{f(K)}(B)$. Since $\pi^Y_{f(K)}(B) \neq C^Y_{nf(K)}$, $C_n(f)(A) = B$. Hence, $C_n(f)$ is onto.

Now, let us prove (4) implies (1). Let $K \in 2^X$ such that $C^n_K(f)$ is onto and $y \in Y$. If $y \in f(K)$, there exists $k_1 \in K$ such that $f(k_1) = y$. Now, suppose that $y \notin f(K)$, $\{y\} \notin C_{nf(K)}(Y)$. Then $\pi^Y_{f(K)}(\{y\}) \neq C^Y_{nf(K)}$. Since $C^n_K(f)$ is onto, there exists $\mathcal{A} \in C^n_K(X)$ such that $C^n_K(f)(\mathcal{A}) = \pi^Y_{f(K)}(\{y\})$. Moreover, there is $A \in C_n(X)$ such that $\pi^X_K(A) = \mathcal{A}$. Since $C^n_K(f) \circ \pi^X_K = \pi^Y_{f(K)} \circ C_n(f)$, $C^n_K(f)(\pi^X_K(A)) = \pi^Y_{f(K)}(C_n(f)(A)) = \pi^Y_{f(K)}(\{y\})$. Thus, there exists $a \in A$ such that $f(a) = y$. □

By Theorem 2 and [12, Proposition 1, p. 784] we have the following result.

Corollary 1. *Let $f\colon X \to Y$ be a mapping between continua and $n \in \mathbb{N}$. Then*
$$C^n_K(f)\colon C^n_K(X) \to C^n_{f(K)}(Y)$$
is onto for every $K \in 2^X$ if and only if f is weakly confluent.

The next example shows us that there are continua X, Y and a mapping $f\colon X \to Y$ such that f is not weakly confluent and $C^n_K(f)$ is onto for some $K \in 2^X$.

Example 1. Let $f\colon I \to S^1$ be defined by $f(t) = (\cos(2\pi t), \sin(2\pi t))$. Then, f is not pseudo-confluent, weakly monotone, or freely decomposable. If $K = \{0\}$, then $C^n_K(f)$ is a monotone mapping for every $n \geq 1$.

Proof. Note that f is not pseudo-confluent, weakly monotone, or freely decomposable. Now, let $K = \{0\}$ and $n \in \mathbb{N}$. We shall prove that $C^n_K(f)$ is monotone. Let $\mathcal{B} \in C^n_{f(K)}(S^1)$. Suppose that $\mathcal{B} = C^{S^1}_{nf(K)}$, by Proposition 5, $C^n_K(f)^{-1}(\mathcal{B})$ is connected. In another case, by Lemma 3,
$$C^n_K(f)^{-1}(C^{S^1}_{nf(K)}) = \pi^I_K\left(\bigcup_{p\in f^{-1}(f(K))} C_{n\{p\}}(I)\right).$$
Then, $C_n(I) - \bigcup_{p\in f^{-1}(f(K))} C_{n\{p\}}(I) = \langle(0,1)\rangle_n$. Since $f|_{(0,1)}$ is one to one, $C^n_K(f)|_{\pi^I_K(\langle(0,1)\rangle_n)}$ is one to one. Therefore, $C^n_K(f)^{-1}(\mathcal{B})$ is connected. □

Example 2. In the interval I, we identify the point 0 with the point $\frac{1}{3}$, and the point $\frac{2}{3}$ with the point 1. Let g be the quotient mapping, note that g is onto and is not weakly confluent. Thus, by [12, Proposition 1, p. 784], $C_n(g)$ is not onto. Moreover, note that for no $K \in 2^X$, $C^n_K(g)$ is onto.

Theorem 3. *Let $f\colon X \to Y$ be a mapping between continua and $n \in \mathbb{N}$. Then the following conditions are equivalent:*
(1) f is a homeomorphism; (2) $C_n(f)$ is a homeomorphism; (3) $C^n_K(f)$ is a homeomorphism for each $K \in 2^X$; (4) $C^n_K(f)$ is a homeomorphism for some $K \in 2^X$.

Proof. By [12, Theorem 46, p. 801] (1) implies (2). Note that (2) implies (3) and (3) implies (4). By Theorem 1 and Theorem 2 f is bijective. Thus, f is a homeomorphism. Therefore (4) implies (1). □

Theorem 4. *Let $f : X \to Y$ be a mapping between continua and $n \in \mathbb{N}$. Consider the following conditions:*
(1) f is a homeomorphism; (2) $C_n(f)$ is open; (3) $C^n_K(f)$ is open for each $K \in 2^X$;
(4) $C^n_K(f)$ is open for some $K \in 2^X$. Then,
(1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3), (1) $\Rightarrow$ (4), (2) $\Rightarrow$ (4), and (3) $\Rightarrow$ (4).

Proof. Clearly each of the conditions (1), (2) or (3) implies (4). By [5, Corollary 3.3, p. 122], (1) and (2) are equivalent. By Theorem 3, we have that (2) implies (3). Now, we prove that (3) implies (2). Let $\mathcal{U}$ be an open subset of $C_n(X)$.

First, we may assume that $X \in \mathcal{U}$. Since $\mathcal{U}$ is an open subset of $C_n(X)$, there exists $\epsilon > 0$ such that $B_\epsilon^{H_d}(X) \cap C_n(X) \subset \mathcal{U}$. Let $0 < \delta < \epsilon$ such that $B_\delta^{H_d}(X) \cap C_n(X) \subset \mathrm{Cl}_{C_n(X)}(B_\delta^{H_d}(X) \cap C_n(X)) \subset \mathcal{U}$. Moreover, using order arcs, it is easy to see that $B_\delta^{H_d}(X) \cap C_n(X)$ is connected. By Proposition 4, there exists $K \in 2^X$ such that

$$C_{nK}(X) \subset \mathrm{Cl}_{C_n(X)}(B_\delta^{H_d}(X) \cap C_n(X)) \subset \mathcal{U}.$$

By [2, Lemma 6.10], $\pi_K^X(\mathcal{U})$ is an open subset of $C_K^n(X)$ containing C_{nK}^X. Since $C_K^n(f)$ is an open mapping, $C_K^n(f)(\pi_K^X(\mathcal{U}))$ is an open subset of $C_{f(K)}^n(Y)$ containing $C_{nf(K)}^Y$. Moreover, note that $C_K^n(f)(\pi_K^X(\mathcal{U})) = \pi_{f(K)}^Y(C_n(f)(\mathcal{U}))$. Thus, $(\pi_{f(K)}^Y)^{-1}(\pi_{f(K)}^Y(C_n(f)(\mathcal{U}))) = C_n(f)(\mathcal{U})$ is an open subset of $C_n(Y)$.

Otherwise, if $X \notin \mathcal{U}$, set $K = X$ then $C_{nK}(X) \cap \mathcal{U} = \varnothing$. Hence, $\pi_K^X(\mathcal{U})$ and $C_K^n(f)(\pi_K^X(\mathcal{U}))$ are open subsets of $C_K^n(X)$ and $C_{f(K)}^n(Y)$, respectively. Note that $C_{nK}^X \notin \pi_K^X(\mathcal{U})$ and $C_{nf(K)}^Y \notin C_K^n(f)(\pi_K^X(\mathcal{U}))$. Since $C_K^n(f)(\pi_K^X(\mathcal{U})) = \pi_{f(K)}^Y(C_n(f)(\mathcal{U}))$, we have

$$(\pi_{f(K)}^Y)^{-1}(\pi_{f(K)}^Y(C_n(f)(\mathcal{U}))) = C_n(f)(\mathcal{U})$$

is an open subset of $C_n(Y)$. Therefore, $C_n(f)$ is an open mapping. □

Example 3. Let $f\colon [-1, 1] \to I$ be the mapping defined by $f(t) = |t|$. Then, f is not a homeomorphism. If $K = \{0\}$, then $C_K^n(f)$ is an open mapping for every $n \geq 1$.

5. Monotone-type mappings. Let $\mathbb{M}$ be any of the following classes of mappings: monotone, almost monotone, quasi-monotone, weakly monotone.

Theorem 5. *Let $f\colon X \to Y$ be a mapping between continua and $n \in \mathbb{N}$. If $C_K^n(f) \in \mathbb{M}$ for each $K \in 2^X$, then $f \in \mathbb{M}$.*

Proof. Let $B \in C(Y) - \{Y\}$ (with $\mathrm{Int}_Y(B) \neq \varnothing$ for the cases: almost monotone, quasi-monotone and weakly monotone). If $f^{-1}(B) = X$, then $C_n(f)(X) = B$. Now suppose that $f^{-1}(B) \subsetneq X$, let $K \in 2^X$ such that $K \cap f^{-1}(B) = \varnothing$. Then $C_n(B)$ is a subcontinuum of $C_n(Y)$ (with $\mathrm{Int}_{C_n(Y)}(C_n(B)) \neq \varnothing$) such that $C_n(B) \cap C_{nf(K)}(Y) = \varnothing$. Thus, we conclude that $\pi_{f(K)}^Y(C_n(B))$ is a subcontinuum of $C_{f(K)}^n(Y) - \{C_{nf(K)}^Y\}$ $(\mathrm{Int}_{C_{f(K)}^n(Y)}(\pi_{f(K)}^Y(C_n(B))) \neq \varnothing)$.

(a) If $C_K^n(f)$ is monotone (or almost monotone), then $C_K^n(f)^{-1}(\pi_{f(K)}^Y(C_n(B)))$ is connected. Since π_K^X is monotone, it follows that $(\pi_K^X)^{-1}(C_K^n(f)^{-1}(\pi_{f(K)}^Y(C_n(B))))$ is connected. Thus, $C_n(f)^{-1}(C_n(B)) = (\pi_K^X)^{-1}(C_K^n(f)^{-1}(\pi_{f(K)}^Y(C_n(B))))$ is connected. Then $C_n(f)^{-1}(C_n(B)) = \langle f^{-1}(B)\rangle_n$. Hence, $f^{-1}(B)$ is connected. Therefore, f is monotone (or almost monotone).

(b) If $C_K^n(f)$ is quasi-monotone (or weakly monotone), then $C_K^n(f)^{-1}(\pi_{f(K)}^Y(C_n(B)))$ has only finitely many components, $\mathfrak{L}_1, \ldots, \mathfrak{L}_m$ such that $C_K^n(f)(\mathfrak{L}_i) = \pi_{f(K)}^Y(C_n(B))$ for each $i \in \{1, \ldots, m\}$. Now let L be a component of $f^{-1}(B)$. By (3) of Lemma 1, $\pi_K^X(\langle L\rangle_n)$ is a component of $C_K^n(f)^{-1}(\pi_{f(K)}^Y(C_n(B)))$. Consequently, each component of $f^{-1}(B)$ determines one component of $C_K^n(f)^{-1}(\pi_{f(K)}^Y(C_n(B)))$. Therefore, $f^{-1}(B)$ has only finitely many components and by (3) of Proposition 2, $f(L) = B$. Hence, f is quasi-monotone (or weakly monotone). □

Theorem 6. *Let $f\colon X \to Y$ be a mapping between continua and $n \in \mathbb{N}$. Then the following conditions are equivalent:*

(1) f is monotone; (2) $C_n(f)$ is monotone; (3) $C^n_K(f)$ is monotone for each $K \in 2^X$. Moreover, each of them implies that
(4) $C^n_K(f)$ is monotone for some $K \in 2^X$.

Proof. By [12, Theorem 4, p.784], (1) and (2) are equivalent. By Proposition 1 and Theorem 5, (2) implies (3) and (3) implies (1), respectively. Clearly, (3) implies (4). □

By Proposition 1 and Theorem 5, we have the following result.

Theorem 7. *Let $f\colon X \to Y$ be a mapping between continua and $n \in \mathbb{N}$. We consider the following conditions:*
(1) $f \in \mathbb{M}$; (2) $C_n(f) \in \mathbb{M}$; (3) $C^n_K(f) \in \mathbb{M}$ for each $K \in 2^X$; (4) $C^n_K(f) \in \mathbb{M}$ for some $K \in 2^X$. Then following implications hold:

(2) ⇒ (3), (2) ⇒ (4), (3) ⇒ (4), (2) ⇒ (1), and (3) ⇒ (1).

Example 1 shows us that there are continua X, Y and a mapping $f\colon X \to Y$ such that f is not monotone, almost monotone, quasi-monotone, or weakly confluent. But $C^n_K(f)$ is monotone for some $K \in 2^X$.

6. Strongly freely decomposable mappings.

Theorem 8. *Let $f\colon X \to Y$ be a mapping between continua and let $n \in \mathbb{N}$. Then, $C^n_K(f)$ is almost monotone if and only if $C^n_K(f)$ is strongly freely decomposable.*

Proof. Suppose that $C^n_K(f)$ is strongly freely decomposable. Since $C^n_K(X)$ is unicoherent (see [2, Theorem 2.1]), by [7, Theorem 4.2, p. 894] $C^n_K(f)$ is almost monotone. Since every almost monotone mapping is strongly freely decomposable, we have proved this theorem. □

The next result follows from Theorem 7 for almost monotone mappings and Theorem 8.

Corollary 2. *Let $f\colon X \to Y$ be a mapping between continua and let $n \in \mathbb{N}$. If $C^n_K(f)$ is strongly freely decomposable, then f is an almost monotone mapping.*

7. Confluent-type mappings. Let $\mathbb{C}$ be any of the following classes of mappings: confluent, semi-confluent, weakly confluent, pseudo-confluent, joining.

Remark 1. Given a continuum X and $n \in \mathbb{N}$. If B is a subcontinuum of X and $x_1, \ldots, x_{n-1} \in X$, then $\mathcal{B} = \langle \{x_1\}, \ldots, \{x_{n-1}\}, B\rangle_n \subset C_n(X)$ is homeomorphic to B. In particular, if B is an irreducible continuum, then $\mathcal{B}$ is an irreducible continuum.

Theorem 9. *Let $f\colon X \to Y$ be a mapping between continua and $n \in \mathbb{N}$. If $C^n_K(f) \in \mathbb{C}$ for each $K \in 2^X$, then $f \in \mathbb{C}$.*

Proof. Let B be a proper subcontinuum (irreducible for the case of pseudo-confluent) of Y. Let D_1 and D_2 be two components of $f^{-1}(B)$. If $f^{-1}(B) = X$, then $C_n(f)(X) = B$. Now suppose that $f^{-1}(B) \subsetneq X$. Note that we can choose $K \in 2^X$ such that $K \cap f^{-1}(B) = \varnothing$ for which there exist $y_1, \ldots, y_{n-1} \in Y - (B \cup f(K))$ such that $y_i \neq y_j$ for $i \neq j$. Let M_i be a component of $f^{-1}(y_i)$ for each $i \in \{1, \ldots, n-1\}$. Then, by (3) of Lemma 1, for each $i \in \{1, 2\}$ $\pi^X_K(\langle M_1, \ldots, M_{n-1}, D_i\rangle_n)$ is a component of $C^n_K(f)^{-1}(\pi^Y_{f(K)}(\mathcal{B}))$ where $\mathcal{B} = \langle \{x_1\}, \ldots, \{x_{n-1}\}, B\rangle_n$. Note that $\mathcal{B} \cap C_{n f(K)}(Y) = \varnothing$ (by Remark 1, this implies that $\pi^Y_{f(K)}(\mathcal{B})$ is a irreducible subcontinuum of $C^n_{f(K)}(Y)$ such that $C^Y_{nf(K)} \notin \pi^Y_{f(K)}(\mathcal{B})$).

(a) If $C_K^n(f)$ is confluent, then by (1) of Proposition 2 for each component D of $f^{-1}(B)$, $C_n(f)(\langle M_1,\ldots,M_{n-1},D\rangle_n)=\mathcal{B}$. Hence, $f(D)=B$. Therefore, f is confluent.

(b) If $C_K^n(f)$ is semi-confluent, without loss of generality we can suppose that

$$C_K^n(f)(\pi_K^X(\langle M_1,\ldots,M_{n-1},D_1\rangle_n))\subset C_K^n(f)(\pi_K^X(\langle M_1,\ldots,M_{n-1},D_2\rangle_n)).$$

Then $C_n(f)(\langle M_1,\ldots,M_{n-1},D_1\rangle_n)\subset C_n(f)(\langle M_1,\ldots,M_{n-1},D_2\rangle_n)$. Thus, $f(D_1)\subset f(D_2)$. Therefore, f is semi-confluent.

(c) If $C_K^n(f)$ is weakly confluent, then by (2) of Proposition 2, there exists a continuum $\mathfrak{M}$ of $C_n(f)^{-1}(\mathcal{B})$ such that $C_n(f)(\mathfrak{M})=\mathcal{B}$. Since $\mathfrak{M}\cap C_{nK}(X)=\varnothing$, we can find a subset M_n of X, such that M_n is a component of $f^{-1}(B)$ and $\langle M_1,\ldots,M_n\rangle_n\cap\mathfrak{M}\neq\varnothing$. By (2) of Lemma 1, $\langle M_1,\ldots,M_n\rangle_n$ is a component of $C_n(f)^{-1}(\mathcal{B})$. Thus, $\mathfrak{M}=\langle M_1,\ldots,M_n\rangle_n$ and $f(M_n)=B$. Therefore f is weakly confluent.

(d) If $C_K^n(f)$ is pseudo-confluent, then there exists a component $\mathfrak{C}$ of $C_K^n(f)^{-1}(\pi_{f(K)}^Y(\mathcal{B}))$ such that $C_K^n(f)(\mathfrak{C})=\pi_{f(K)}^Y(\mathcal{B})$. Since $C_{nK}^X\notin\mathfrak{C}$, it follows that $(\pi_K^X)^{-1}(\mathfrak{C})$ is a component of $(\pi_K^X)^{-1}(C_K^n(f)^{-1}(\pi_{f(K)}^Y(\mathcal{B})))=C_n(f)^{-1}(\mathcal{B})$. Note that $C_n(f)((\pi_K^X)^{-1}(\mathfrak{C}))=\mathcal{B}$.
On the other hand, by [15, Lemma 1, p. 1578], $\bigcup(\pi_K^X)^{-1}(\mathfrak{C})$ has at most n components. But $\bigcup(\pi_K^X)^{-1}(\mathfrak{C})\subset f^{-1}(y_1)\cup\cdots\cup f^{-1}(y_{n-1})\cup f^{-1}(B)$. Moreover, $f^{-1}(y_i)\cap(\bigcup(\pi_K^X)^{-1}(\mathfrak{C}))\neq\varnothing$ for each $i=1,\ldots,n-1$ and $(\bigcup(\pi_K^X)^{-1}(\mathfrak{C}))\cap f^{-1}(B)\neq\varnothing$. Then $\bigcup(\pi_K^X)^{-1}(\mathfrak{C})$ has exactly n components, let's say $C_1,\ldots,C_n$. Without loss of generality, we assume that $C_i\subset f^{-1}(y_i)$ for $i=1,\ldots,n-1$ and $C_n\subset f^{-1}(B)$. Let C be the component of $f^{-1}(B)$ such that $C_n\subset C$.
Claim. $f(C)=B$.
Let $b\in B$ and let $E=\{y_1,\ldots,y_{n-1}\}\cup b$. Then $E\in\mathcal{B}$. Hence, there exists $A\in(\pi_K^X)^{-1}(C)$ such that $f(A)=E$. This implies that $b\in f(A)\subset f(\bigcup(\pi_K^X)^{-1}(\mathfrak{C}))=f(C_1)\cup\cdots\cup f(C_n)$. If there exists $j\in\{1,\ldots,n-1\}$ such that $b\in f(C_j)$, then $b=y_j$, this is a contradiction. Hence, $b\in f(C_n)$. Thus, $b\in f(C)$. It follows that $B\subset f(C)$. Therefore, f is pseudo-confluent.
(e) If $C_K^n(f)$ is joining, then

$$C_K^n(f)(\pi_K^X(\langle M_1,\ldots,M_{n-1},D_1\rangle_n))\cap C_K^n(f)(\pi_K^X(\langle M_1,\ldots,M_{n-1},D_2\rangle_n))\neq\varnothing.$$

Thus, $C_n(f)(\langle M_1,\ldots,M_{n-1},D_1\rangle_n)\cap C_n(f)(\langle M_1,\ldots,M_{n-1},D_2\rangle_n)\neq\varnothing$ and, in consequence, $f(D_1)\cap f(D_2)\neq\varnothing$. Therefore, f is joining. □

By Proposition 1 and Theorem 9, we have the following result.

Theorem 10. *Let $f\colon X\to Y$ be a mapping between continua and $n\in\mathbb{N}$. We consider the following conditions:*
(1) $f\in\mathbb{C}$; (2) $C_n(f)\in\mathbb{C}$; (3) $C_K^n(f)\in\mathbb{C}$ for each $K\in 2^X$; (4) $C_K^n(f)\in\mathbb{C}$ for some $K\in 2^X$.
Then following implications hold:

(2) ⇒ (3), (2) ⇒ (4), (3) ⇒ (4), (2) ⇒ (1), and (3) ⇒ (1).

Example 1 shows us that there are continua X, Y and a mapping $f\colon X\to Y$ such that f is not pseudo-confluent, weakly confluent, semi-confluent, or confluent, but $C_K^n(f)$ is confluent for some $K\in 2^X$.

7. OM, atriodic and light mappings. Let X be a continuum. Given a sequence $\{A_m\}_{m\in\mathbb{N}}$ of nonempty subsets of X we define $\limsup_{m\to\infty}A_m$ as the set of points $x\in X$ such that there exists a sequence of positive integers $m_1<m_2<\cdots$ and points $x_{m_s}\in A_{m_s}$ such that $\lim x_{m_s}=x$.

Lemma 4. ([12, Lemma 12, p. 788]) *A mapping $f\colon X \to Y$ between continua is OM if and only if for each point $y \in Y$ and each sequence of points $y_m \in Y$ converging to y, the set $\limsup_{m\to\infty} f^{-1}(y_m)$ meets each component of $f^{-1}(y)$.*

Theorem 11. *Let $f\colon X \to Y$ be a mapping between continua and $n \in \mathbb{N}$. We consider the following conditions:*
(1) f is OM; (2) $C_n(f)$ is OM; (3) $C^n_K(f)$ is OM for each $K \in 2^X$; (4) $C^n_K(f)$ is OM for some $K \in 2^X$. Then following implications hold:

$$(2) \Rightarrow (3),\ (2) \Rightarrow (4),\ (3) \Rightarrow (4),\ (2) \Rightarrow (1), \text{ and } (3) \Rightarrow (1).$$

Proof. Clearly, (2) implies (1) and (3) implies (4). By Proposition 1, (2) implies (3). Set $y \in Y$. Let $\{y_i\}_{i\in\mathbb{N}}$ be a sequence of points in Y converging to y. We consider $K \in 2^X$ such that $f(K) \cap \{y, y_1, \ldots\} = \varnothing$. Take $z_1, \ldots, z_{n-1} \in Y - (f(K) \cup \{y, y_1, \ldots\})$ such that $z_j \neq z_l$ for $j \neq l$. Let M_j be a component of $f^{-1}(z_j)$ for each $j \in \{1, \ldots, n-1\}$ and let M_n be a component of $f^{-1}(y)$. By (3) of Lemma 1, $\pi^X_K(\langle M_1, \ldots, M_n\rangle_n)$ is a component of $C^n_K(f)^{-1}(\pi^Y_{f(K)}(\{z_1, \ldots, z_{n-1}, y\}))$. Since the sequence $\{\pi^Y_{f(K)}(\{z_1, \ldots, z_{n-1}, y_i\})\}_{i\in\mathbb{N}}$ converges to $\pi^Y_{f(K)}(\{z_1, \ldots, z_{n-1}, y\})$, by Lemma 4,

$$\pi^X_K(\langle M_1, \ldots, M_n\rangle_n) \cap \limsup_{t\to\infty} C^n_K(f)^{-1}(\pi^Y_{f(K)}(\{z_1, \ldots, z_{n-1}, y_t\})) \neq \varnothing.$$

Let $A \in \langle M_1, \ldots, M_n\rangle_n$ be such that $\pi^X_K(A) \in \limsup_{t\to\infty} C^n_K(f)^{-1}(\pi^Y_{f(K)}(\{z_1, \ldots, z_{n-1}, y_t\}))$. Then, there exists a subsequence $\{\pi^X_K(A_{t_r})\}_{r\in\mathbb{N}}$ such that for each $r \in \mathbb{N}$, $\pi^X_K(A_{t_r}) \in C^n_K(f)^{-1}(\pi^Y_{f(K)}(\{z_1, \ldots, z_{n-1}, y_{t_r}\}))$ and $\lim_{r\to\infty} \pi^X_K(A_{t_r}) = \pi^X_K(A)$. Let $a \in A \cap M_n$. Since $\lim_{r\to\infty} A_{t_r} = A_n$, there exists a sequence $\{a_{t_r}\}_{r\in\mathbb{N}}$, with $a_{t_r} \in A_{t_r}$, such that it converges to $a \in A$. Thus, there exists a positive integer m_0 such that $f(a_{t_r}) = y_{t_r}$ for each $r \geq m_0$. Hence $a \in A \cap M_n \cap \limsup_{t\to\infty} f^{-1}(y_t)$. Therefore, by Lemma 4, f is OM. □

The mapping $f\colon X \to Y$ of Example 1 is not OM but $C^n_K(f)$ is OM for some $K \in 2^X$.

Theorem 12. *Let $f\colon X \to Y$ be a mapping between continua and $n \in \mathbb{N}$. We consider the following conditions:*
(1) f is atriodic; (2) $C_n(f)$ is atriodic; (3) $C^n_K(f)$ is atriodic for each $K \in 2^X$; (4) $C^n_K(f)$ is atriodic for some $K \in 2^X$. Then following implications hold:

$$(2) \Rightarrow (3),\ (2) \Rightarrow (4),\ (3) \Rightarrow (4),\ (2) \Rightarrow (1), \text{ and } (3) \Rightarrow (1).$$

Proof. Note that (2) implies (1) and (3) implies (4). By Proposition 1, (2) implies (3). Let B be a proper subcontinuum of Y. If $f^{-1}(B) = X$, then $C_n(f)(X) = B$. Now suppose that $f^{-1}(B) \subsetneq X$, let $K \in 2^X$ such that $K \cap f^{-1}(B) = \varnothing$. We consider $y_1, \ldots, y_{n-1} \in Y - (B \cup f(K))$ with $y_i \neq y_j$ for $i \neq j$. Set $\mathcal{B} = \langle \{y_1\}, \ldots, \{y_{n-1}\}, B\rangle_n$. Since $C^n_K(f)$ is an atriodic mapping, there exist two components $\mathfrak{D}_1$ and $\mathfrak{D}_2$ of $C^n_K(f)^{-1}(\pi^Y_{f(K)}(\mathcal{B}))$ such that:

(a) $C^n_K(f)(\mathfrak{D}_1) \cup C^n_K(f)(\mathfrak{D}_1) = \pi^Y_{f(K)}(\mathcal{B})$,

(b) for each component $\mathfrak{C}$ of $C^n_K(f)^{-1}(\pi^Y_{f(K)}(\mathcal{B}))$, we have either $C^n_K(f)(\mathfrak{C}) = \pi^Y_{f(K)}(\mathcal{B})$, or $C^n_K(f)(\mathfrak{C}) \subset C^n_K(f)(\mathfrak{D}_1)$ or $C^n_K(f)(\mathfrak{C}) \subset C^n_K(f)(\mathfrak{D}_2)$.

For each $j = 1, 2$, we have that $(\pi^X_K)^{-1}(\mathfrak{D}_j) \cap C_{nK}(X) = \varnothing$. Then there exist $M^j_1, \ldots, M^j_n$ of X such that M^j_i is a component of $f^{-1}(y_i)$ for each $i = 1, \ldots, n-1$ and M^j_n is a component of $f^{-1}(B)$. We may assume that $\mathfrak{D}_j \cap \pi^X_K(\langle M^j_1, \ldots, M^j_n\rangle_n) \neq \varnothing$. Since $\mathfrak{D}_j$ is a component of $C^n_K(f)^{-1}(\pi^Y_{f(K)}(\mathcal{B}))$, by (3) of Lemma 1, $\mathfrak{D}_j = \pi^X_K(\langle M^j_1, \ldots, M^j_n\rangle_n)$. Thus, by **(a)**, $f(M^1_n) \cup f(M^2_n) = B$. Now, let C be a component of $f^{-1}(B)$. Since $\pi^X_K(\langle M^1_1, \ldots, M^1_{n-1}, C\rangle_n)$ is a component of $C_K^n(f)^{-1}(\pi_{f(K)}^Y(\mathcal{B}))$, by **(a)**, we have either $f(C) = B$, or $f(C) \subset f(M_n^1)$ or $f(C) \subset f(M_n^2)$. □

Theorem 13. *Let $f\colon X \to Y$ be a mapping between continua and $n \in \mathbb{N}$. We consider the following conditions:*
(1) f is light; (2) $C_n(f)$ is light; (3) $C_K^n(f)$ is light for each $K \in 2^X$; (4) $C_K^n(f)$ is light for some $K \in 2^X$. Then following implications hold:

$$(2) \Rightarrow (1),\ (3) \Rightarrow (1),\ \text{and } (3) \Rightarrow (4).$$

Proof. Clearly, (3) implies (4). It follows from [11, Theorem 3.10, p. 185] that (2) implies (1). Now, suppose that $C_K^n(f)$ is a light mapping. To prove that f is a light mapping, we may assume that exists $y \in Y$ such that $f^{-1}(y)$ is not totally disconnected. Note that $f^{-1}(y) \neq X$, in the contrary case, $C_K^n(f)$ is a constant mapping. Now, let M be a nondegenerate component of $f^{-1}(y)$. Let $K \in 2^X$ such that $K \cap f^{-1}(y) = \varnothing$ and let $y_1, \dots, y_{n-1} \in Y - (f(K) \cup \{y\})$ such that $y_i \neq y_j$ for $i \neq j$. Let M_i be a component of $f^{-1}(y_i)$ for each $i = 1, \dots, n-1$. By (3) of Lemma 1, $\pi_K^X(\langle M_1, \dots, M_{n-1}, M\rangle_n)$ is a subcontinuum nondegenerate of $C_K^n(f)^{-1}(\pi_{f(K)}^Y(\{y_1, \dots, y_{n-1}, y\}))$, this is a contradiction. □

Example 4. Let $f\colon [-1,1] \to I$ be the mapping defined by $f(t) = |t|$. Then, f is light. If $K = \{1\}$, then $C_K^n(f)$ is not light for every $n \geq 1$.

Proof. Since $f^{-1}(f(K)) = \{-1, 1\}$, by Lemma 3, $C_K^n(f)^{-1}(C^I_{nf(K)})$ is nondegenerate. By Proposition 5, $C_K^n(f)^{-1}(C^I_{nf(K)})$ is a connected subset of $C_K^n([-1,1])$. Therefore, $C_K^n(f)$ is not light. □

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Universidad Autónoma del Estado de México, Facultad de Ciencias,
Instituto Literario No. 100, Col. Centro, C.P. 50000, Toluca, Estado de México, México
jgao@uaemex.mx
eca@uaemex.mx
jamartinezc@uaemex.mx

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